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# Harmonic and transient scattering from weakly nonlinear objects

Dan Censor [1]

Department of Electrical and Computer Engineering
and Biomedical Engineering and Science Institute
Drexel University

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A mathematical model for scattering of electromagnetic waves from weakly nonlinear objects is developed. The constitutive relations are based on Volterra series, but additional, physically plausible, heuristic assumptions have to be introduced in order to solve the scattering problem The general theory is discussed in connection with scattering from circular cylinders. These canonical problems demonstrate the new phenomena involved. It is shown that the first order effects of the nonlinear scattering problem involve modification of the linear scattering coefficients and production of new multipole terms at the fundamental frequency In addition, part of the energy is transformed into harmonics. The corresponding problem of transient scattering is considered The new effects of pole migration and pole creation are discussed. The present study contributes to understanding the theoretical aspects of nonlinear scattering, and may also provide a method for remote sensing of nonlinear targets.

## INTRODUCTION

A general review of nonlinear wave propagation [Korpel and Banerjee, 1984] provides a good link to the existing literature. The analysis of weakly nonlinear wave systems, using constitutive relations based on the Volterra series has been considered recently [Caspers, 1964; Schubert and Wilhelmi, 1971; Akhmanov and Kokhlov, 1972; Censor, 1976, 1983, 1985a], sometimes without giving due credit to Volterra's [1959] original work on functional series, which has been introduced into nonlinear systems and communication theory by Wiener [1958],[Bedrosian and Rice, 1971; Bussgang et al., 1974; Dalpe et al., 1982] See Schetzen [1980] for additional fundamental theory and references Essentially, the Volterra series is the functional analog of the Taylor series, hence this approach works well when the physical situation justifies the truncation of the series after a few leading terms For weakly nonlinear systems, defined below, the model is adequate. It is therefore a limited tool but still systematic, not simply an *ad hoc* method. Hence it facilitates examination and discussion of fundamental assumptions. See relevant remarks by Bedrosian and Rice [1971]

In the present study the Volterra series constitutive relations are taken as the starting point. For the harmonic scattering problem the scatterer is excited by a single frequency, hence all frequencies involved are the fundamental $\omega$ and harmonics $p\,\omega$, $|\,p\,| = 1,2,3....$ In the nonlinear medium we therefore have a periodic rather than a harmonic wave. Weak nonlinearity is characterized by the condition of phase matching [Censor, 1976, 1983, 1985a]. This means that all harmonics must have the same phase velocity as the fundamental wave If this condition is realized, all the local interactions act coherently, giving rise to a significant harmonic production. On the other hand, in the absence of phase matching, contributions of nonlinear interaction are not in phase and tend to disappear because of destructive interference. By definition, in weakly nonlinear systems the ensuing waves are of negligible amplitude. The phase matching implies that shocks cannot develop in the system, because such a shape change of the waves can only occur if harmonics propagate at different phase velocities. Consequently it is assumed that the present periodic wave does not change its shape as it moves from one part of the medium to another. A question which is often asked at this point is: But suppose we are dealing with a dispersive medium, then how is it possible for different

[1] Visiting Professor and Louis and Bessie Stein Family Foundation Fellow, on leave from the Department of Electrical and Computer Engineering, Ben Gurion University of the Negev, Beer Sheva, Israel



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frequencies to propagate with the same phase velocity? This question is based on notions from the linear regime. In the nonlinear case the dispersion equation, relating $\omega$ and the propagation vector $\mathbf{k}$, provided such a relation can be defined, must involve the amplitudes of the fields. This means that the phase velocity of the waves at harmonic frequencies is determined by their amplitudes, and the amplitudes are such that phase matching takes place. The interdependence of dispersion and nonlinearity is a well known phenomenon, and the fact that nonlinearity can counteract dispersion facilitates the creation of solitons [Korpel and Banerjee, 1984]. In essence, we are saying that the periodic waves of the present problem are solitons.

Going one step further, it seems plausible to assume that in weakly nonlinear media, interaction of waves propagating in different directions can be ignored, because, (unless the medium is anisotropic and special precautions are taken) the nonlinear interaction products are non phase-matched with respect to the fundamental wave. This facilitates the use of superposition of waves, in a restricted sense as explained below. This is also consistent with the fact that solitons admit to superposition [Korpel and Banerjee, 1984]. This approach facilitates the construction of wave functions from integrals of plane waves, in order to satisfy the boundary conditions relevant to the scattering problems in question. Up to a certain point the present study follows the previous argument [Censor, 1983], but here a technique of approximating the first order nonlinear effects is introduced, yielding explicit results for scattering by cylinders

It is shown that in the presence of nonlinearity the scattering coefficients of the limiting linear case are modified, and that new multipoles are created. The problems are extremely complicated, and in order to obtain some explicit results, only the first order nonlinear correction terms are derived.

The problem of transient scattering, essentially the pulse response of a scatterer, has been studied for the linear case [Baum, 1976a, b, 1978; Dolph and Scott, 1978; Bennett and Ross, 1978, Miller and Landt, 1980], and its relation to the harmonic response has been considered [Marin, 1973, 1974; Dolph and Cho, 1980, Ramm, 1980]. The impulse response of a general nonlinear system is not available. The closest one can get is the computation of kernels in the integrals appearing in the Volterra series [Bedrosian and Rice, 1971; Schetzen, 1980], but this is not quite the same. In the context of weakly nonlinear systems discussed here we are able to investigate the effects. It is argued that nonlinearity affects the impulse response by causing migration and creation of poles. Inasmuch as the pole configuration is a signature of the specific scatterer, in principle the nonlinear effects of pole migration and pole creation can be used to identify nonlinear scatterers

Before turning to the theory itself, it should be emphasized that, in order to facilitate the analytical discussion, a lot of heuristic assumptions are introduced below In the future it will be necessary to perform suitable laboratory experiments to establish or refute the validity of the present model The fundamental theory has been considered previously [Censor, 1976, 1983, 1985a; Volterra, 1959] and for completeness will be succinctly summarized here. We consider Maxwell's equations for the electromagnetic field in sourceless domains, and nonlinear constitutive relations $\mathbf{D}(\mathbf{E})$, $\mathbf{B}(\mathbf{H})$, as sums of terms, a hierarchy with the leading terms being paramount. $\mathbf{D}=\mathbf{D}^{(1)}+\mathbf{D}^{(2)}+\cdots$. Combining space-time variables $x_i, t$, $i=1,2,3$ into a four-vector $\mathbf{X}$, the Volterra series are given by $\mathbf{D}=\sum_{i,n} D_i^{(n)}(\mathbf{X})$, where

$$D_i^{(n)}(\mathbf{X})=\int d^4\mathbf{X}_1\cdots\int d^4\mathbf{X}_n$$
$$[\epsilon_{i,j,\cdots,\nu}^{(n)}(\mathbf{X}_1,\cdots,\mathbf{X}_n)E_j(\mathbf{X}-\mathbf{X}_1)\cdots E_\nu(\mathbf{X}-\mathbf{X}_n)] \quad (1)$$

which can be considered as the generalization of the convolution integral, hence Volterra systems include dispersion effects. Substituting a periodic plane wave solution $E_i=\sum_p E_{pi}\, e^{ip\theta}$, $\theta=\mathbf{K}\,\mathbf{X}=\mathbf{k}\,\mathbf{x}-\omega t$ where $\theta,\mathbf{k},\omega,\mathbf{K}$ are the phase, propagation vector, (angular) frequency, and the $k_i,\omega$ combined into a four-vector $\mathbf{K}$, respectively, (1) becomes

$$D_i^{(n)}=\sum_p E_{pj}\cdots$$
$$\sum_{p'} E_{p'\nu}\, e^{i(p+\cdots+p')\theta}\epsilon_{ij\cdots\nu}(p\,\mathbf{K},\cdots,p'\,\mathbf{K}) \quad (2)$$

When dealing with nonlinear problems, which involve powers of the fields, the complex exponential must be adequately qualified. Some authors prefer to use the exponential notation and carefully add the complex conjugate, e.g., $E_{pi}\, e^{ip\theta}+(E_{pi}\, e^{ip\theta})^*$. This is understood without further elaboration in (2) and throughout. Hence if $\mathbf{E}$ is real, $D_i^{(n)}=\sum_\gamma D_{\gamma i}^{(n)} e^{i\gamma\theta}$ is real and periodic too.

Terms can be regrouped subject to $\gamma=p+\cdots+p'$ yielding

$$D_{\gamma i}^{(n)}=\overline{\epsilon}_{ij\cdots\nu}^{(n)}(\gamma\mathbf{K})E_{\gamma j}\cdots E_{\gamma\nu}$$
$$\overline{\epsilon}_{ij\cdots\nu}^{(n)}=\sum_{p,\cdots p'}\frac{E_{pj}}{E_{\gamma j}}\cdots\frac{E_{p'\nu}}{E_{\gamma\nu}}\epsilon_{ij\cdots\nu}^{(n)}(p\,\mathbf{K},\cdots,p'\,\mathbf{K}) \quad (3)$$

The original $\epsilon^{(n)}$ in (1) are considered to be input independent, hence true constitutive parameters of the system The new $\overline{\epsilon}^{(n)}$ of (3) depend on ratios of amplitudes, hence they are input dependent, in principle. This is the price we have to pay for algebraization of the constitutive relations. Many studies dealing with nonlinear system implicitly assume that algebraic constitutive relations as in (3) are valid, and treat $\overline{\epsilon}^{(n)}$ as constants. We too shall adopt this approach, but not as an *ad hoc* heuristic assumption, observing that the constancy of $\bar{\epsilon}^{(n)}$ is at best an approximation. This heuristic assumption is tantamount to asserting that ratios of amplitudes of various harmonics are amplitude independent. For weakly nonlinear systems this seems to be a plausible assumption. In other words, the weakly nonlinear $D_i(\mathbf{E})$ is a perturbation near $\mathbf{E} = 0$, and expanding it in a Taylor series yields

$$D_i(\mathbf{E}) = D_i(0) + \frac{\partial D_i}{\partial E_j}\Bigg|_0 E_j + \cdots + \frac{1}{n!}\frac{\partial^n D_i}{\partial E_j \cdots \partial E_\nu}\Bigg|_0 (E_j \cdots E_\nu) + \cdots \tag{4}$$

With $D_i(0) = 0$, comparison of (3) and (4) establishes $\bar{\epsilon}^{(n)}$ as amplitude independent parameters in the limit of small signals.

Finally, substituting periodic waves and constitutive relations (3) into Maxwell's equations yields, for each harmonic

$$\begin{aligned}(\mathbf{k}\times\mathbf{E})_i &- \omega\bar{\mu}_{ij}^{(1)}H_j \\ - \omega\bar{\mu}_{ijk}^{(2)}H_j H_k &- \omega\bar{\mu}_{ijkl}^{(3)}H_j H_k H_l - \cdots = 0 \\ (\mathbf{k}\times\mathbf{H})_i &+ \omega\bar{\epsilon}_{ij}^{(1)}E_j \\ + \omega\bar{\epsilon}_{ijk}^{(2)}E_j E_k &+ \omega\bar{\epsilon}_{ijkl}^{(3)}E_j E_k E_l + \cdots = 0\end{aligned} \tag{5}$$

where $p$, indicating the number of the harmonic, is suppressed

For $|p| = 1$, the fundamental harmonic, the six scalar equations (5) will have nontrivial solutions if the determinant vanishes. Since this statement usually applies to linear algebra, its meaning in the present context must be explained. Successive substitution of equations in (5) yields [Censor, 1985a] one equation in which $E_i$ or $H_i$ for a specific index $i$ is a factor. If this field component is nonvanishing, then the expression multiplying it must vanish. This expression (which can be obtained in various equivalent forms) is the associated amplitude dependent dispersion relation $F(\mathbf{k},\omega,\mathbf{E},\mathbf{H}) = 0$. For inhomogeneous time varying media, under suitable restrictions [Censor, 1976], we have $F(\mathbf{k},\omega,\mathbf{E},\mathbf{H},\mathbf{x},t) = 0$.

## CYLINDRICAL WAVES IN NONLINEAR MEDIA

Inasmuch as we allow superposition of non phase-matched plane waves, and conditions for existence of plane periodic waves have been stated, it is possible to define new wave functions for weakly nonlinear media in the form

$$\mathbf{E} = \hat{\mathbf{z}}\int_C \sum_p \bar{E}_p(\tau) e^{ipk(\tau) r\cos(\psi-\tau) - ip\omega t}\, d\tau/2\pi \tag{6}$$

where $\hat{\mathbf{z}}$ is the direction of the cylindrical axis, the contour of integration $C$ is yet undetermined and $\bar{E}_p(\tau)/2\pi$ is the amplitude of the wave propagating in the direction defined by the (in general complex) azimuthal angle $\tau$. For the problem of the circular cylinder $\bar{E}_p(\psi)$ is recast as a Fourier series

$$\bar{E}_p(\psi) = \sum_n a_{pn}\, e^{in\psi} \tag{7}$$

The coefficients $a_{pn}$ will be determined by the pertinent boundary conditions The range of $n$ in (7) corresponds to the multipole structure of the field. The field $\mathbf{E}$ and its mate $\mathbf{H}$ must satisfy Maxwell's equations. The validity of (6) is therefore restricted to cases where a field E can exist in the $\hat{\mathbf{z}}$ direction alone. In the following discussion an isotropic medium is considered, hence (6) exists, and the mate field $\mathbf{H}$ can be represented in a form similar to (6), oriented perpendicularly to $\hat{\mathbf{z}}$ with an appropriate $\bar{\mathbf{H}}$ in (6) replacing $\bar{\mathbf{E}} = \bar{E}\hat{\mathbf{z}}$. It follows from (5)-(7) and the appropriate dispersion equation $F(\mathbf{k},\omega,\bar{\mathbf{E}},\bar{\mathbf{H}}) = 0$ that $k$ too is a function of directions $\psi$, hence in (6),$k$ must be replaced by $k(\tau)$. This raises the question of expressing (6) in terms of the already familiar special functions, e g., in the linear case with constant $k$, (6) can be recast in terms of Bessel functions. There is no general answer to this question, however, subject to certain restrictions, essentially deriving first order correction terms to the known linear cases, (6) can again be recast in terms of Bessel functions.

Let us demonstrate this on a simple example, taking into account $p = 1$ only, i.e., the fundamental harmonic. The nonlinearity is confined to the electric field, i e., $\bar{\mu}_{ij}^{(1)} \neq 0$, but all higher order tensors $\bar{\mu}_{i\cdots\nu}^{(n)}$, $n > 1$ vanish, and $\bar{\mu}^{(1)} = \mu$ is chosen as a constant scalar. Further simplifying the problem, $\bar{\epsilon}_{ij}^{(1)}$ will be taken as a constant scalar $\epsilon^{(1)}$, and the nonlinear effects will be introduced through $\bar{\epsilon}_{ijkl}^{(3)}$, taken as a constant scalar $\epsilon^{(3)}$ Thus an isotropic medium is defined. Eliminating $\mathbf{H}$ in (5) yields

$$(\mathbf{k}\times\mathbf{k}\times\mathbf{E})_i + \omega^2\mu\epsilon^{(1)}E_i + \omega^2\mu\epsilon^{(3)}E_i^{\,3} = 0 \tag{8}$$

This medium supports transversal waves, and from (6), (8), the dispersion equation for this case is

$$F = k^2 - \omega^2\mu(\epsilon^{(1)} + \epsilon^{(3)}\bar{E}^2) = 0 \tag{9}$$

which is insensitive to the sign of $\bar{E}$. To the first order in $\epsilon^{(3)}$, (9) is approximately

$$k = k^{(1)} + \kappa,\ k^{(1)} = \omega(\mu\epsilon^{(1)})^{1/2},\ \kappa = \frac{k^{(1)}\epsilon^{(3)}}{2\epsilon^{(1)}}\bar{E}^2$$

$$\kappa = \sum_\nu k_\nu e^{i\nu\psi} = \frac{k^{(1)}\epsilon^{(3)}}{2\epsilon^{(1)}}\sum_{n,n'} a_n a_{n'}\, e^{i(n+n')\psi} \tag{10}$$

where $\kappa$ is recast as a Fourier series, displaying the dependence of $k$ on directions $\psi$, and the fact that $k_\nu$

are corrections terms dependent on the small parameter $\epsilon^{(3)}$. In view of the orthogonality of the exponentials we obtain

$$k_\nu = \frac{k^{(1)}\epsilon^{(3)}}{2\epsilon^{(1)}} \sum_{n,\, n'=\nu-n} a_n\, a_{n'} \qquad (11)$$

where only terms $\nu = n + n'$ appear. Substituting (10), (11) in (6) and using the approximation $e^{\alpha} = 1+\alpha$ for small $\alpha$, we get

$$\mathbf{E} = \hat{\mathbf{z}} e^{-i\omega t} \int_C e^{ik^{(1)}r\cos(\psi-\tau)}\{[1+ir\cos(\psi-\tau)\sum_\nu k_\nu e^{i\nu\tau}] \sum_n a_n\, e^{in\tau}\} d\tau/2\pi \qquad (12)$$

The expression in braces in (12) can be recast as

$$[1 + \frac{ir}{2}\sum_\nu (e^{i\psi}k_{\nu+1} + e^{-i\psi}k_{\nu-1})e^{i\nu\tau}]\sum_n a_n\, e^{in\tau} \qquad (13)$$

with indices judiciously raised and lowered. This clearly demonstrates the coupling between modes, because according to (10) products of coefficients $a_n$ now appear, and if the range of $n$ was finite originally, it is now extended, due to $n+\nu$. For example, if in the linear limiting case the range is $n=0,\pm1$, i.e., $a_0, a_1, a_{-1} \neq 0$, then $\nu$ will have the range $\nu = 0,\pm1,\pm2$ and $n+\nu$ in (13) will have the range $0,\pm1,\pm2,\pm3$. Note, however that $n=0$ alone corresponds to $\nu=0$, hence will not produce higher order multipoles. The lowest term to show effects of nonlinearity in the multipoles is the dipole term $n=0,\pm1$. It will be shown below that these effects, taking place in the interior of the nonlinear scatterer, also affect the scattered wave. This provides a method for remote sensing the nonlinear properties of the scatterer.

Usually the interior of the scatterer will be the nonlinear region, therefore $C$ in (6) will be chosen as the real Sommerfeld contour for the nonsingular Bessel functions [Stratton, 1941], hence (12), (13) can be recast as

$$\mathbf{E} = \hat{\mathbf{z}} e^{-i\omega t}\{\sum_n i^n\, a_n\, J_n(k^{(1)}r)e^{in\psi} + \frac{r}{2}\sum_{n,\nu} i^{(n+\nu+1)} a_n\, J_{n+\nu}(k^{(1)}r) [k_{\nu+1}e^{i(n+\nu+1)\psi} + k_{\nu-1}e^{i(n+\nu-1)\psi}]\}$$
$$= \hat{\mathbf{z}} e^{-i\omega t}\sum_n i^n\, e^{in\psi}\{a_n\, J_n(k^{(1)}r) + \frac{rk_0}{2}\sum_\nu [a_{n-\nu-1}J_{n-1}(k^{(1)}r) - a_{n-\nu+1}J_{n+1}(k^{(1)}r)]\} \qquad (14)$$

where the second expression is obtained by appropriately raising and lowering indices and $k_0$ is given by (11) for $\nu=0$, i.e., summation over all $n+n'=0$. The mate $\mathbf{H}$ field is obtained from the Maxwell equations (5) upon substitution of (14).

## HARMONIC SCATTERING BY NONLINEAR CYLINDERS

We are now adequately equipped to study the problem of scattering of a harmonic plane electromagnetic wave by a nonlinear circular cylinder. As a concrete example, the dielectric, isotropic, nonlinear medium considered above is adopted, existing in the interior of the cylinder, $0 \le r \le \rho$. The incident and scattered waves have their $\mathbf{E}$ field polarized along the cylindrical axis, hence they are given by

$$\hat{\mathbf{z}} e_0 e^{ikx-i\omega t} = \hat{\mathbf{z}} e_0 \sum_n i^n\, J_n(kr)e^{in\psi-i\omega t}$$
$$\hat{\mathbf{z}}\sum_n i^n\, b_n\, H_n(kr)e^{in\psi-i\omega t} \qquad (15)$$

respectively, where $k$ is pertinent to the external medium $r > \rho$; $H_n = H_n^{(1)}$ are the Hankel functions of the first kind, which together with $e^{-i\omega t}$ generate outgoing waves, and $b_n$ are the coefficients we seek. There are also higher harmonics present, but we deal here with the fundamental harmonic only. The field inside the cylinder $r \le \rho$ is given by (14). In electromagnetic theory [Stratton, 1941] boundary conditions are usually obtained from Maxwell's equations $\nabla\times\mathbf{E} = -\partial_t\mathbf{B}$, $\nabla\times\mathbf{H} = \partial_t\mathbf{D}$, $\nabla\cdot\mathbf{D} = \nabla\cdot\mathbf{B} = 0$ without reference to the constitutive relations $\mathbf{B}(\mathbf{H})$, $\mathbf{D}(\mathbf{E})$ Therefore we have the same boundary conditions as in the linear case, namely the continuity of the tangential $\mathbf{E}$ and $\mathbf{H}$ fields across the boundary $r=\rho$. In view of the orthogonality of the functions $e^{in\psi}$, and subject to the boundary conditions, we obtain

$$e_0 J_n(k\rho) + b_n H_n(k\rho) - a_n J_n(k^{(1)}\rho) - \frac{\rho k_0}{2}\sum_\nu [a_{n-\nu-1}J_{n-1}(k^{(1)}\rho) - a_{n-\nu+1}J_{n+1}(k^{(1)}\rho)] = 0$$
$$e_0 J_n'(k\rho) + b_n H_n'(k\rho) - \frac{k^{(1)}}{k}a_n J_n'(k^{(1)}\rho) - \frac{k^{(1)}\rho k_0}{2k}\sum_\nu [a_{n-\nu-1}J_{n-1}'(k^{(1)}\rho) - a_{n-\nu+1}J_{n+1}'(k^{(1)}\rho)] = 0 \qquad (16)$$

where the primes denote differentiation of the Bessel functions with respect to the argument, and it is assumed that $\mu$ has the same value everywhere The zero order approximation, i e , the linear case, is obtained by setting $\epsilon^{(3)} = 0$ and solving (16). This yields the well known result

$$a_n^{(1)} = e_0 \frac{J_n(k\rho)H_n'(k\rho) - J_n'(k\rho)H_n(k\rho)}{J_n(k^{(1)}\rho)H_n'(k\rho) - \frac{k^{(1)}}{k}J_n'(k^{(1)}\rho)H_n(k\rho)}$$
$$b_n^{(1)} = e_0 \frac{\frac{k^{(1)}}{k}J_n(k\rho)J_n'(k^{(1)}\rho) - J_n'(k\rho)J_n(k^{(1)}\rho)}{J_n(k^{(1)}\rho)H_n'(k\rho) - \frac{k^{(1)}}{k}J_n'(k^{(1)}\rho)H_n(k\rho)} \qquad (17)$$

The next step is to substitute (17) into (16), into all the terms involving $k_0$, because they have $\epsilon^{(3)}$ as a factor By inspection of (10), (16), (17) it is clear that these terms have $e_0^3$ as a factor. We therefore rewrite (16) in the form

$$e_0 J_n(k\rho) + e_0^3 f_n(\rho) + b_n H_n(k\rho) - a_n J_n(k^{(1)}\rho) = 0$$

$$e_0 J_n'(k\rho) + e_0^3 g_n(\rho) + b_n H_n'(k\rho) - \frac{k^{(1)}}{k} a_n J_n'(k^{(1)}\rho) = 0 \quad (18)$$

where $f_n(\rho)$, $g_n(\rho)$ are obtained by regrouping terms involving (17) and are therefore known functions, and solve for $a_n$, $b_n$ .This yields (17) with $J_n(\rho)$ replaced by $J_n(k\rho)+e_0^2 f_n(\rho)$ and $J_n'(k\rho)$ replaced by $J_n'(k\rho)+e_0^2 g_n(\rho)$ hence the problem is solved. Note that the range of $n$ in (17) and in the coefficients $a_n$ ,$b_n$ derived from (18) is not identical, because of the mode interaction which produces multipoles of higher order, as explained above. The new coefficients obtained from (18) depend nonlinearly on $e_0$ and are therefore sensitive to the amplitude of the incident wave This is characteristic of the nonlinear scattering problem and can serve to identify nonlinear properties of scatterers

The solution of the scattering problem for the case of the **H** field polarized along the axis, and for the case of normal incidence but arbitrarily oriented **E** field needs special consideration It is shown here that for a combination of **E** and **H** fields polarized along $\hat{\mathbf{z}}$ there exists a nonlinear interaction, hence such modes cannot be superposed. First consider the case of normal incidence with the **H** field polarized along $\hat{\mathbf{z}}$ i.e., **E** field transversal to the cylindrical axis. Similarly to (6), (7), we write

$$\mathbf{H} = \hat{\mathbf{z}} e^{-i\omega t} \int e^{ik(\tau) r \cos(\psi-\tau)} \overline{H}(\tau) d\tau / 2\pi$$

$$\overline{H}(\psi) = \sum_n a_n' e^{in\psi} \quad (19)$$

According to (5), the linear part of the mate electric field is given by $\overline{\mathbf{E}} = \mathbf{k}\times\hat{\mathbf{z}}\overline{H}/\omega\epsilon^{(1)}$ and we have

$$\overline{\mathbf{E}}\,\overline{\mathbf{E}} = (k^2/\omega^2 \epsilon^{(1)\,2})\overline{H}^2 \quad (20)$$

Inasmuch as we seek expressions correct to first order in $\epsilon^{(3)}$, the linear part of $\overline{H}^2$ can be substituted in (9), yielding

$$F = k^2 - k^{(1)\,2} - (\omega^2\mu^2\epsilon^{(3)}) \sum_{n,n'} a_n' a_{n'}' e^{i(n+n')\psi} = 0 \quad (21)$$

and consistent with our approximation this yields

$$k = k^{(1)} + \frac{k^{(1)}\epsilon^{(3)}\mu}{2\epsilon^{(1)\,2}} \sum_{n,n'} a_n' a_{n'}' e^{i(n+n')\psi}$$

$$= k^{(1)} + \kappa = k^{(1)} + \sum_\nu k_\nu e^{i\nu\psi} \quad (22)$$

By substitution of $k$ from (22) to (19) and expansion of the exponential as in (12) etc , we finally derive (14). The analog of (15)-(18) follows upon replacing $e_0$ by $h_0 = e_0(\epsilon/\mu)^{1/2}$ the amplitude of the incident magnetic field, replacing $a_n$ ,$b_n$ ,$f_n$ ,$g_n$ by $a_n'$ ,$b_n'$ ,$f_n'$ ,$g_n'$ , and $k^{(1)}/k$ by $\epsilon k^{(1)}/(\epsilon^{(1)}k)$ .The problem can therefore be considered solved. The case of normal incidence ( **k** vector perpendicular to $\hat{\mathbf{z}}$ ,as before) with $\overline{\mathbf{E}}$ arbitrarily oriented leads to nonlinear interaction in $\overline{\mathbf{E}}\cdot\overline{\mathbf{E}}$ in (9). For this case we have $(\overline{\mathbf{E}}_= + \overline{\mathbf{E}}_|)\cdot(\overline{\mathbf{E}}_= + \overline{\mathbf{E}}_|) = |\overline{\mathbf{E}}_=|^2 + |\overline{\mathbf{E}}_||^2$ where =, | denote the components of the **E** field parallel and perpendicular to $\hat{\mathbf{z}}$ ,respectively, and $\overline{\mathbf{E}}_=^2$, $\overline{\mathbf{E}}_|^2$ are given by $\alpha_=^2 \sum_{n,n'} a_n a_{n'} e^{i(n+n')\psi}$, $(\epsilon\alpha_|^2/\mu \sum_{n,n'} a_n' a_{n'}' e^{i(n+n')\psi}$ respectively, where $\alpha_= = \mathbf{e}_0\,\hat{\mathbf{z}}$, $\alpha_| = |\mathbf{e}_0\times\hat{\mathbf{z}}|$ and $\mathbf{e}_0$ is the electric field amplitude of the incident plane wave. The $k$ thus obtained is now used instead of (10) to derive the scattered **E** field polarized along $\hat{\mathbf{z}}$ , and instead of (22) in deriving the scattered field **H** polarized along $\hat{\mathbf{z}}$ . The remaining mate fields are finally obtained from Maxwell's equations (5).

Associated with the waves at the fundamental frequency are waves with frequencies $p\omega$ The amplitudes $\overline{\mathbf{E}}_p$ ,$\overline{\mathbf{H}}_p$ are determined by $\mathbf{k}$,$\omega$ of the fundamental harmonic, obtained from (5) for $p = 1$ and by the equations (5) pertinent to the harmonic $p$ in question. Usually the constitutive tensors for higher harmonics are not available, hence this part of the problem will be discussed in general terms only. Although there is no external excitation at these frequencies, their existence in the interior of the scatterer and the continuity of the tangential fields prescribed by the boundary conditions, leads to fields at harmonic frequencies in the scattered wave. The fact that nonlinear scatterers respond by radiating waves at harmonic frequencies provides another method for identifying such objects.

## TRANSIENT SCATTERING BY NONLINEAR OBJECTS

Transient scattering, i.e., the response of obstacles to incident sharp pulses is of great interest, providing a method for remote sensing of objects by means of their natural (i.e., resonance) complex frequencies. Obviously the new phenomena introduced by nonlinearity provide additional information on the geometry and composition of scatterers. It is therefore of interest to relate the results, obtained above for harmonic excitation, to the case of pulse excitation.

Linear transient scattering is reviewd by numerous authors [Baum, 1976a, b, 1978; Dolph and Scott, 1978; Bennett and Ross, 1978; Miller and Landt, 1980], and the relation to harmonic scattering is also considered in the literature [Marin, 1973, 1974; Dolph and Cho, 1980, Ramm, 1980] For simplicity, consider the incident plane wave to be a unit impulse

$$\mathbf{e}_0\delta(t - \frac{\hat{\mathbf{k}}\,\mathbf{r}}{c}) = \frac{\mathbf{e}_0}{2\pi}\int_{-\infty}^{\infty} e^{\,i\omega(t-\hat{\mathbf{k}}\,\mathbf{r}/c)}d\,\omega \qquad (23)$$

The teansient response is obtained by integrating the scattered harmonic field over all frequencies

$$\mathbf{u}(\mathbf{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\,\omega\, \mathbf{u}_\omega(\mathbf{r})e^{-i\omega t} = \sum \text{residues} \qquad (24)$$

where $\mathbf{u}_\omega(\mathbf{r})e^{-i\omega t}$ is the scattered wave at frequency $\omega$ and the integral can be evaluated as the sum of residues of the complex poles. These complex poles correspond to the complex resonance frequencies for which the coefficients of the eigenmodes, e.g., equation (17), become singular. This happens when the denominator vanishes while the numerator is finite. The associated pole map can be considered as the characteristic signature for the object at hand and provides a remote sensing and identification method. The ringing response of the scatterer can be represented as

$$\mathbf{u}(\mathbf{r},t) = \sum_n W_n(k_n\,\mathbf{r})e^{-i\omega_n t}$$

$$k_n = \omega_n/c\,, \quad t > 0 \qquad (25)$$

where $W_n$ are solutions of the vector Helmholtz wave equation and $\omega_n$ are the complex frequencies, i.e., the poles in the complex $\omega$ plane. Methods for extracting $\omega_n$ from signals, also in the presence of noise, are currently intensively investigated, but this aspect is outside the scope of the present study.

In order to discuss nonlinear transient scattering, the question of self interaction of spectral components of the incident pulse must be resolved. Recalling that the discussion is confined to weak nonlinear systems, it follows that interaction of plane waves in an integral like (23) exists only if the waves are phase matched. In the case of waves as in (6), (15), it has been argued that the plane waves are not phase matched in an isotropic medium on account of their different directions of propagation. In the present case, a spectral component of frequency $\omega$ will excite harmonics $p\,\omega$ at frequencies already existing in the incident pulse, hence there is a reshuffling of the budget of spectral components From (3) it follows that the tensors $\overline{\epsilon}^{(n)}$, $\overline{\mu}^{(n)}$, because of their dependence on amplitudes, depend on the shape of the incident pulse. This will modify the parameters appearing in (5), but will not change the nature of the problem or invalidate our results. Hence if $\epsilon^{(3)}(\omega)$ for the problem at hand is considered to be known as a function of frequency, and the integration (24) is performed, the singularities associated with the solution of (18) will be effective. The same effect is expected for the corresponding three-dimensional problem. The effect of nonlinearity on the scatteting coefficients will cause pole migration in the complex $\omega$ plane, and the appearance of new multipole terms gives rise to creation of new poles. If we know how to read the new pole map, then we can learn about the nonlinear properties of the scatterer

## CONCLUDING REMARKS

Previous studies have been devoted to study the general problem of propagation of electromagnetic waves in weakly nonlinear media, and have also mentioned the scattering problem. Presently the scattering problem is studied in some detail. The analysis, in terms of circular cylindrical wave functions, has been approached as a perturbation scheme, in order to derive first order corrections to the limiting linear case. For cylindrucal waves results are derived in detail. For the three-dimensional vector wave problems detailed computations are too cumbersome. Although explicit results have not been derived, the step by step method described above facilitates the derivation if specific problems of interest arise.

The new phenomena characterizing nonlinear scattering are the production of harmonics, which has been mentioned but not studied in detail, the appearance of nonlinear effects in the scattering coefficients and the nonlinearity induced mode coupling. The last phenomenon also means that the range of multipoles is extended, if originally this range was finite in the linear problem The corresponding transient scattering problem involves pole migration and pole creation in the complex $\omega$ plane. A special aspect of the transient scattering problem is the fact that the residues and pole locations may be affected by the shape of the exciting pulse.

Recently, first-order velocity effects on harmonic and transient scattering has been discussed [Censor, 1984, 1985b, 1986]. There too, first order effects produced pole migration and creation, although the pertinent formulas have different forms. In a way, this is not surprising, because time dependent obstacles produce new frequencies, due to the modulation of the incident wave by the moving surface of the scatterer, and this can also be construed as a nonlinear effect.

A vast body of literature exists to date, discussing nonlinear propagation problems In spite of the progress made, there is no general framework emerging, and probably there will never be Hence we have to satisfy ourselves with *ad hoc* results obtained for special classes of problems. In the present case, although some explicit results are obtained, more analytical and experimental studies will be necessary in order to establish their validity.

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D. Censor, Drexel University, Biomedical Engineering and Science Institute, Philadelphia, PA 19104.