

Theory of the Doppler effect: Fact, fiction and approximation

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The theory of the generalized Doppler effect, concerning the scattering of waves from arbitrary time-varying objects, is still a wide open class of problems. Over the years, two major approaches have been used for the analysis of the Doppler effect: (1) the "Einstein recipe" of coordinate transformations and (2) the direct solution of the time-dependent boundary value problem. In addition, numerous approximate methods have been devised. Some approaches, e.g., the so-called quasi-stationary method, are inherently inconsistent, as explained here. As long as uniform rectilinear motion is involved, we stand on relatively firm ground, and various methods and approximations can be compared and evaluated. For nonuniform motion in the presence of plane waves and plane interfaces (including the logical extension to geometrical optics), some results are available, but the problem already involves heuristic assumptions. Some special problems have been solved for two- and three-dimensional configurations, under very restrictive conditions. The general problem of scattering by a time-varying obstacle is still an open problem. A delineation of some methods for solving such problems numerically is given and discussed.

INTRODUCTION AND SUMMARY

As early as 1842, C. J. Doppler published his original article {Doppler, 1842}, and since then the term "Doppler effect" became a household item in numerous branches of science and engineering. The original work on this fascinating phenomenon, concerning the color of double stars, is nowadays of historical interest only. The literature concerned with applications of the Doppler effect mushroomed to such an extent that a comprehensive review paper in journal article form is practically im-

possible. Dipping into a data pool of engineering and scientific publications with the phrase "Doppler effect" yields thousands of titles. Divorcing ourselves from the aspect of applications and narrowing our interest to the questions of the theory cuts down the number considerably. However, even in the realm of theory there exists a diversity of subjects, hence the scope of the present study must be further specified. Excluded from the present discussion are medium effects {Gill, 1965}, although they are of great interest, ranging from applications to propagation in moving media, to the exotic theory of the complex Doppler effect {Papadopoulos, 1965}.

What will be studied here is the Doppler effect theory pertaining to scattering of waves from moving obstacles. In this class of problems two major approaches

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have been used over the years. The first, called here "Einstein's recipe," prescribes transformations of fields from a given ("laboratory") system of reference, to the rest ("comoving") system of reference of the scatterer. In the latter the configuration of the scatterer is time independent, allowing the application of known methods to solve the scattering problem. The results are then transformed back to the laboratory system. In an overoptimistic statement, Einstein {1905} declares: "All problems in the optics of moving bodies can be solved by the method here employed. What is essential is, that the electric and magnetic force" {the electromagnetic fields} "of the light which is influenced by a moving body, be transformed into a system of coordinates at rest relatively to the body. By this means all problems in the optics of moving bodies will be reduced to a series of problems in the optics of stationary bodies." This prescription fails for cases where different parts of the object move with different velocities and for cases where transformations are not available, e.g. due to nonuniform motion. The second method, in those cases where it is applicable, is to solve the time dependent boundary value problem directly. The question of the relevance of the problems involving nonuniform motion is controversial. While the present author considers it to be a central issue, an anonymous reviewer commented that boundary conditions in any frame are themselves macroscopically averaged approximations consequent of equation of motion of microscopic bounding materials. Hence all moving body problems involving classical electrodynamics in vacuum flat space-time are fully formulated and solvable by means of Einstein's special relativity theory.

As long as we deal with plane reflectors and normal incidence, i.e., one dimensional wave problems, the problem is relatively simple, due to the existence of the D'Alembert solution {Censor, 1973a}. The one dimensional problem is reviewed for uniform motion. A related method, the so-called "quasi-stationary method," is reexamined. It is shown that it does

not follow as a limiting case of the exact methods. For nonuniform motion the question of using the instantaneous velocity in expressions established for constant velocity is examined. It is claimed that the validity of this approximation is much more limited than what is usually assumed. The direct solution of problems in the laboratory system is discussed, and the dependence of the method on the existence of the D'Alembert solution is emphasized. For two- and three-dimensional problems involving uniform rectilinear motion the Einstein recipe is applicable. For concreteness, a specific formalism based on plan waves integrals is summarized. However, for multiple scattering the Einstein recipe works only if a successive scheme is considered. Laboratory system solutions for two-dimensional problems are feasible only for the case of planar reflectors and oblique incidence of a plane wave. The problem can be slightly generalized by representing an arbitrary excitation as a superposition of plane waves. However, a general method for solving two- and three-dimensional problems in the laboratory system is nonexistent. Some approximate solutions for periodically moving objects are also mentioned.

Finally, the question of solving problems involving arbitrary configurations and velocity fields is considered. Two general classes of problems are feasible, provided the instantaneous velocity approximation is applicable. For short wavelengths, the theory for oblique incidence and plane reflectors can be extended to cases admitting geometrical optics considerations. For objects whose dimensions are on the scale of the wavelength, it is suggested that the original object be replaced by a collection of small scatterers, e.g., thin cylinders, small spheres, for two- and three-dimensional problems, respectively. The solution will then be found by using a successive scattering scheme. These methods involve numerical computations and have not been tried out yet.

By way of apology, the author wishes to stress that the present discussion reflects some personal preferences, especially in areas close to his own research

activity. However, every effort has been made to list all literature sources relevant to the present subject.

ONE-DIMENSIONAL PROBLEMS

Uniform motion. This is the simplest class of problems and can be traced back to the original results of Doppler [1842]. Einstein [1905] perceived the importance of the simple problem of scattering of a plane wave by a uniformly moving plane reflector and analyzed it in his original work. The correct formulas for reflection by a moving mirror have been given by Abraham [1904], before the advent of Einstein's theory of special relativity; see also von Laue [1965]. For a while, this problem played a central role in physics, being used to compare the new theory to classical results [Pauli, 1958]. It has been shown by many researchers [von Mosengeil, 1907; Harnack, 1912; Titow, 1925] over a long period of time, that nonrelativistic considerations lead to the same results obtained for the above problem from relativistic formulas. Not surprisingly, corresponding results are also obtained for acoustical waves. Similar concepts of energy and radiation pressure can also be applied to the two cases, yielding identical formulas [Censor and Schoenberg, 1971].

For concreteness, consider the electromagnetic problem for normal incidence of a plane wave on a uniformly moving reflector [Einstein, 1905]. Define

$$\begin{aligned} e_i e^{ik_i x - i\omega_i t} \\ e_r e^{-ik_r x - i\omega_r t} \end{aligned} \quad (1)$$

the incident and reflected waves, respectively, with indices i,r denoting incident, scattered quantities, respectively. The reflector moves according to

$$x = a + vt \quad (2)$$

With the boundary condition that the total tangential electric field at the boundary vanishes (in the comoving system of

reference in which the boundary is at rest), the observer in the laboratory system of reference measures

$$\frac{e_r}{e_i} = -e^{i(k_i + k_r)a} \frac{\omega_r}{\omega_i}$$

$$\frac{\omega_r}{\omega_i} = \frac{k_r}{k_i} = \frac{1 - \beta}{1 + \beta} \approx 1 - 2\beta$$

$$\beta = v/c < 1, \quad c = \omega_i/k_i = \omega_r/k_r = (\mu\epsilon)^{-1/2} \quad (3)$$

These results are readily obtained by applying the space-time transformations and the transformations pertinent to the electromagnetic fields, as prescribed by Einstein [1905; Pauli, 1958]. For this simple problem, relativistically exact results can be obtained by nonrelativistic arguments. This is essentially what Abraham and others did. Thus the sum of waves in (1) must satisfy some yet undetermined boundary condition, symbolized by  $\sigma$ , at the time-dependent boundary whose position is defined by (2), i.e.,

$$\begin{aligned} \sigma \{ e_i e^{ik_i a - i\omega_i t(1 - \beta)} \\ + e_r e^{-k_r a - i\omega_r t(1 + \beta)} \} = 0 \end{aligned} \quad (4)$$

Hence for arbitrary t this prescribes  $\omega_r/\omega_i, k_r/k_i$  as in (3). To derive the equations for the amplitudes, i.e., the boundary condition, we have to use an elaborate argument based on energy considerations, which, now that the Einstein theory is well established, sounds cumbersome and heuristic. Consider the time-averaged Poynting vector flux [Harnack, 1912], entering and leaving the reflector (which will be assumed to be a lossless perfect mirror). The energy per unit cross section entering the reflector during 1 sec. is  $e_i^2(c - v)/(2\eta)$ , where  $\eta = (\mu/\epsilon)^{1/2}$ , and  $c - v$  takes into account the fact that because of the mo-

tion the reflector evades part of the energy carried by the wave (which would otherwise be absorbed by a fixed reflector). Similarly, we have  $e_r^2(c+v)(2\eta c)$  for the energy emitted by the reflector per second per unit cross section area. The difference between the two quantities is dissipated by the radiation pressure [Pauli, 1958; Stratton, 1941; Panofski and Phillips, 1962] acting on the reflector and performing work along the distance  $v$ : hence by invoking energy conservation,

$$\frac{1}{2\eta} (e_i^2(1-\beta) - e_r^2(1+\beta)) = pv = p_{op_0} v \quad (5)$$

where  $p_0$ , the radiation pressure for a reflector at rest is equal to the total density at the reflector,  $p_0 = e_i^2/(c\eta)$ . The ratio  $p/p_0$  is argued to be

$$\frac{p}{p_0} = \frac{1-\beta}{1+\beta} \quad (6)$$

because this expression satisfies many limiting cases: for  $\beta \ll 1$ ,  $p = p_0$  and  $p_0 v$  is the correct energy converted to mechanical work by an infinitely slow reflector; for  $\beta \rightarrow 1$  the reflector avoids the incident wave, hence no reflected wave is produced and  $p/p_0 \rightarrow 0$  is a plausible result. For  $\beta \rightarrow -1$  we get  $p/p_0 \rightarrow \infty$ , which is plausible in view of the fact that no reflected wave can be emitted, and an infinite amount of energy is accumulated by the moving reflector. Of course, no such lengthy argument is needed once Einstein's theory is accepted. On the other hand, this demonstrates the difficulties encountered in cases where special relativity cannot be applied.

The quasi-stationary method. A few researchers use or mention this approach [Kleinman and Mack, 1979; Cooper, 1980a,b; Van Bladel and De Zutter, 1981] which is attractive because of its simplicity. However, it is inherently inconsistent, and therefore its use without due reservations might lead to erroneous results. The essential idea

is to take into account the time-varying position of the reflector, assuming that the position is occupied at each point by a reflector at rest. Thus the position of the reflector is defined by means of  $a(t)$ . The incident wave (1) is shifted to the location of the reflector by means of

$$\xi = x - a(t) \quad (7)$$

The response to the incident wave

$$\begin{aligned} e_i e^{ik_i a(t)} e^{ik_i \xi - i\omega_i t} & \text{ is a reflected wave} \\ -e_i e^{ik_i a(t)} e^{-ik_i \xi - i\omega_i t} & = \\ = -e_i e^{2ik_i a(t)} e^{-ik_i x - i\omega_i t} & \quad (8) \end{aligned}$$

This result can be interpreted as a wave with an instantaneous frequency  $\omega_r$  given by

$$\omega_r = -\frac{\partial}{\partial t} (2k_i a(t) - \omega_i t) = \omega_i - 2k_i \frac{\partial a}{\partial t} \quad (9)$$

In the case of uniform motion  $a(t) = a_0 + vt$  we obtain

$$\omega_r = \omega_i (1 - 2\beta) \quad (10)$$

which happens to be a correct first order in  $\beta$  approximation to the frequency. But, the following points must be noted: (1) equation (8) is not a proper plane wave satisfying the wave equation  $(\nabla^2 - c^{-2}\partial^2/\partial t^2)\psi(x,t) = 0$ , and the discrepancy is of order  $\beta$ , which is the order of the approximation. In other words, the approximation is not consistent in that it does not yield all correction terms of order  $\beta$ ; (2) another way of considering the result is to note that  $\omega_i/k_i \neq \omega_r/k_r \neq c$ , i.e., the two waves, propagating in the same medium, do not satisfy the same dispersion equation, as required; (3) in some studies this is taken into account heuristically, but the boundary conditions are still inadequate, terms of order  $\beta$  are still missing; (4) although for normal incidence it does not appear, the quasi-stationary method does not display fundamental properties

of velocity dependent reflection, such as aberration of the reflected wave; and (5) yet another way of perceiving this method is to identify (7) for  $a(t) = vt$  as a Galilean transformation and add to it the Galilean time invariance. However, it is still well known that the application of Galilean transformations to Maxwell's equations involves discrepancies of order  $\beta$ . There is nothing wrong in applying Galilean transformations to acoustics, if the wave equation is also adequately transformed [Censor and Schoenberg, 1971]. This is not the case in the quasi-stationary method. Moreover, the use of a Galilean transformation, tacitly or explicitly, leads to results which do not account for time retardation. This "action at a distance" is a serious flaw inherent in the method. Again, some authors have introduced "brute force" corrections to overcome this difficulty.

The question of instantaneous velocity.  
From the previous subsections it is evident that the main source of difficulties is the analysis of nonuniform motion. Many studies attempted to circumvent this difficulty by postulating the validity of the Einstein formulas of special relativity for instantaneous velocities. The use of global transformations of this kind has not yet been satisfactorily justified. Many researchers simply put forward the postulate, although others are more careful, e.g., Brandstatter [1963, p. 236] who remarks (see also references cited there): "The legitimacy of using what is substantially a Lorentz transformation of the field quantities, for accelerated media is an assumption whose validity must ultimately rest on experimental verification." This, of course, is the scientific method in a nutshell. Since experimental evidence is lacking, the use of this method justifies the provocative title of the present study. It is interesting to note that some attention has been given in the literature to infinitesimal successive Lorentz transformations [Møller, 1952]. The present problem of nonuniform motion clearly ties in with this subject. It is therefore imperative that

the problem of using instantaneous Lorentz transformations be considered in future studies, because of its fundamental nature.

To avoid the complexity and intricacy, also the mystique of the theory of relativity, consider the problem of transformations in the context of quasi-Galilean transformation given below, which for low velocities approximates the Lorentz transformation. For  $a(t) = vt$  with constant  $v$ , equation (7) constitutes a Galilean transformation, approximating the Lorentz transformation to the first order in  $\beta$ . It relates the laboratory location  $x$  with the comoving location  $\xi$  by means of the distance  $vt$  covered by the moving origin  $\xi = 0$ . More generally (7) relates  $x, \xi$  when the motion is arbitrary, and the motion of the origin  $\xi = 0$  is given instantaneously by

$$\frac{dx(t)}{dt} = \frac{da(t)}{dt} = v(t) \quad (11)$$

However, what is done by people postulating the validity of the Lorentz transformation for instantaneous velocities, is to replace (7) by a transformation of the form

$$\xi = x - v(t)t \quad (12)$$

which evidently does not follow from (7), (11). One should rather use the integral form

$$\xi = x - \int_0^t v(t)dt \quad (13)$$

consistent with (7), (11). For (13) to conform with (12) we required  $tdv/dt = 0$ . A favorite statement when applying the instantaneous velocity Lorentz transformation is to say the acceleration is negligible. But mathematically, in view of  $tdv/dt \neq 0$ , such a statement is meaningless. There is of course a pragmatic reason for using such a dubious "approximation," we simply do not know a better way of solving the problem. However, as the title suggests, newcomers to the field will be better served if in all honesty, fact, fiction, and approximation are stated

as such. A first-order approximation to the time in the Lorentz transformation prescribes

$$\tau = t - vx/c^2 \quad (14)$$

For constant  $v$ , (14) yields a consistent first-order approximation for the Doppler effect (3). In trying to extend (14) to variable  $v(t)$ , we are confronted with the same difficulties. Rewriting (14) in the form

$$c\tau = ct - vx/c \quad (15)$$

facilitates an interpretation (within a first order in  $\beta$  approximation) in terms of retardation of signals. Thus the signal (light in free space) covers a distance  $ct$  in  $t$  seconds to reach an observer at rest, while an observer moving toward the source will register the signal after it covers a distance  $c\tau$ . At time  $\tau = t = 0$  the moving observer was at  $x = 0$ , together with the fixed observer. The length  $c(t - \tau)$  is equal to the distance covered by the moving observer until it met the oncoming signal. This equals the velocity of the moving observer, multiplied by the time  $x/c$  for the signal to cover the distance  $x$ . Now, if the velocity is not a constant, say we know  $v(x)$  as a function of the position of the moving observer, then it would seem plausible to replace  $vx/c$  of (15) by

$\frac{1}{c} \int_0^x v(x) dx$ ; hence (14) becomes

$$\tau = t - \frac{1}{c} \int_0^x v(x) dx \quad (16)$$

It must be emphasized that the present review does not aspire to innovate generalized Lorentz transformations, merely emphasize the heuristic nature of using the instantaneous velocity in the Lorentz transformation in a simplistic manner. The subject of properly extending the Lorentz transformation to arbitrary motion is out of the scope of the present discussion [Post and Bakulikar, 1971; Ryff, 1975; Kerner, 1976]. It suffices to note that the question of using the instanta-

neous velocity in the Lorentz transformation [Mo, 1970] is not yet resolved. The solution of the Doppler problem according to (13), (16), let alone some pseudorelativistic version thereof, has not been studied, as far as we know. It is not clear what will happen to the notorious invariance of Maxwell's equations subject to such a transformation, hence the question of determining correct boundary conditions for nonuniformly moving objects is yet unanswered. It seems that the best one can do at the moment, although in some cases this lies in the realm of fiction, or to be less literary, is completely heuristic, is to use the instantaneous velocity in the Lorentz transformation only if the change of velocity during a cycle of the signal is small.

A laboratory system method based on the D'Alembert solution. The lack of a reliable transformation in the case of nonuniform motion suggests [Censor, 1973a] that direct methods in the laboratory system be sought. An analysis of the string problem with a rolling support demonstrates the feasibility of the approach for one dimensional problems [Censor, 1973a]. It must be noted, however, that this does not resolve the problem of the boundary conditions that should be used in the electromagnetic case; also, the method is restricted to the one-dimensional wave problem, because the D'Alembert solution exists for this case only. Recently, the method to be described has been applied to one-dimensional electromagnetic problems [Cooper, 1980b; Van Bladel and De Zutter, 1981]; see also Censor [1981], and response by Van Bladel and De Zutter.

Briefly, we assume incident and reflected waves, represented by D'Alembert solutions for the one-dimensional wave equation. The total wave is given by

$$y = f(t - x/c) + g(t + x/c) \quad (17)$$

The boundary condition in the laboratory system, for the string problem say, is taken as  $y = 0$  at  $x = X(t)$ , the time-

dependent location of the boundary.  
Hence at the boundary

$$f(t - X(t)/c) + g(t + X(t)/c) = 0$$

or

$$g(\zeta) = -f\{\zeta - 2X(t)/c\}$$

$$\zeta = t + X(t)/c \quad (18)$$

But, on the assumption that we can solve for  $t = T(\zeta)$ , (18) can be written as

$$g(\zeta) = -f\{\zeta - 2X(T(\zeta))/c\} \quad (19)$$

Now, we know that the general form of the reflected wave is  $g(t + x/c)$ , therefore we substitute  $\zeta = t + x/c$  in (19), obtaining

$$g(t + x/c) = -f\{t + x/c - 2X(T(t + x/c))/c\} \quad (20)$$

This solution satisfies the wave equation, and by substituting  $x = X(t)$  into (20), we fall back on (18), i.e., the boundary condition is satisfied, hence, this is the sought solution. Of course, the inversion  $t = T(\zeta)$  might be very complicated, but in general the method is viable.

Vibrating reflectors. This class of problems has been discussed recently {Cooper, 1980b; Van Bladel and De Zutter, 1981; Censor, 1973a; Knowles, 1967; Liu, 1971; Borkar and Yang, 1973, 1975} but has also been considered from time to time before, particularly the question of harmonically moving reflectors. The problem is of interest for the theory of the Doppler effect, but has been also motivated by various engineering applications. Still, a lot of heuristics is involved, and a general theory has not yet emerged.

TWO AND THREE DIMENSION: UNIFORM MOTION

Plane reflector, oblique incidence.

The reflection problem given by Einstein {1905} was more general than indicated above, and included oblique

incidence, for a reflector moving in free space. Slightly more general formulas are obtained when phase velocity in material media is considered {Censor, 1969}. For free space, the oblique incidence problem has been analyzed in the laboratory system of reference {Abraham, 1904; von Laue, 1965; Pauli, 1958; von Mosengeil, 1907; Harnack, 1912; Titow, 1925} yielding results identical to those obtained from special relativity. Thus Einstein {1905} obtained (say the electric field is in the plane of incidence and let  $a = 0$  in (2))

$$\frac{e_r}{e_i} = \frac{\omega_r}{\omega_i} = \frac{k_r}{k_i} = \frac{1 - 2\beta \hat{k}_i \cdot \hat{v} + \beta^2}{1 - \beta^2}$$

$$\hat{k}_r \cdot \hat{v} = \frac{(1 + \beta^2) \hat{k}_i \cdot \hat{v} - 2\beta}{1 - 2\beta \hat{k}_i \cdot \hat{v} + \beta^2} \quad (21)$$

$\hat{k}_i, \hat{k}_r, \hat{v}$  are unit vectors in the direction of incidence, reflection and velocity, respectively. The latter is perpendicular to the plane of the reflector. Writing the incident and reflected waves as

$$e_i e^{ik_{ix}x + ik_{iy}y - i\omega_i t}$$

$$e_r e^{ik_{rx}x + ik_{ry}y - i\omega_r t} \quad (22)$$

substituting  $x = vt$ , and imposing boundary conditions yields (21). The boundary condition, as in the normal incidence case, can be determined from an energy argument. For the boundary condition to be satisfied for all  $y, t$ , the new Snell law

$$k_{iy} = k_{ry} \quad (23)$$

and

$$k_{ix}v - \omega_i = k_{rx}v - \omega_r \quad (24)$$

must be satisfied. The relations (21) follow, It is worthwhile to note that the aberration phenomenon, i.e.,

$\hat{k} \cdot \hat{v} \neq \hat{k}' \cdot \hat{v}$  for nonvanishing velocity is a first order effect in  $\beta$ . It is inconsistent to label it as a "relativistic effect," and then neglect it by arguing that nonrelativistic approximations are performed. Such an approach is inconsistent because the results (21) can be obtained from (23), (24) without resort to special relativity theory. Methods which do not preserve the proper aberration effects, e.g., the quasi-stationary method, are therefore questionable, what we call here fiction, as opposed to approximation proper. We will return to this point in connection with nonuniform motion.

Scattering by moving objects.

Einstein's recipe has been applied to many uniformly moving objects for which the solution of the velocity independent problem is available {Lee and Mitra, 1967; Tsandoulas, 1968, 1969; Shiozawa, 1968; Restrck, 1968; Ott and Hufford, 1968; Lang, 1971; Twersky, 1971a, b; Censor, 1971, 1972a; Hunter, 1972, 1973a; b; LeVine, 1973; De Zutter and Van Bladel, 1977}. Besides providing solutions to interesting applied problems, these studies give a mathematical formulation to the Einstein recipe and demonstrate its feasibility for two- and three-dimensional problems, for which the D'Alembert solution in its simple form does not exist. Thus Censor ({1972a} earlier presented in his dissertation (1967) in Hebrew) uses plane wave representations, Doppler amplitude operators {Censor, 1969}, and makes ample use of the so-called "invariance of the phase."

For simplicity, the free space case will be summarized. Denoting comoving system quantities by apostrophe, the incident plane wave in the comoving system,  $f' e^{i\psi'}$ , is obtained from the laboratory-given field  $f e^{i\psi}$  by using

$$\begin{aligned} f' &= \tilde{F} \cdot f \\ \tilde{F} &= ((1 - \gamma) \hat{y} + \gamma \beta \hat{k}) \hat{y} + \gamma(1 - \beta \hat{y} \cdot \hat{k}) \tilde{I} \\ \tilde{f} &= \underline{E}, \underline{H}, \quad \tilde{f}' = \underline{E}', \underline{H}' \\ \psi' &= \underline{k}' \cdot \underline{r}' - \omega' t' = \psi = \underline{k} \cdot \underline{r} - \omega t \\ \gamma &= (1 - \beta^2)^{-1/2} \end{aligned} \tag{25}$$

where  $\tilde{I}$  is the idemfactor dyadic. The so-called "invariance of the phase"  $\psi = \psi'$ , which simply means a substitution of the Lorentz transformation and regrouping of terms, prescribes

$$\begin{aligned} \underline{k}' &= \{\tilde{I} - \hat{y}((1 - \gamma)\hat{y} + \gamma\beta\hat{k})\} \cdot \underline{k} \\ \omega' &= \omega\gamma(1 - \beta\hat{y} \cdot \hat{k}) \end{aligned} \tag{26}$$

Further confining the problem to cylindrical scatterers of arbitrary cross section, with the excitation field polarized along the cylindrical axis and the velocity perpendicular to it, we obtain a scalar problem. Thus the scattered field  $u'(r', t')$ , measured in the comoving system, outside the circular cylinder enclosing the scatterer, can be represented as

$$\begin{aligned} u'(r', t') &= \sum_{m=-\infty}^{\infty} i^m a_m(\omega') H_m(k' r') e^{im\theta' - i\omega' t'} \end{aligned} \tag{27}$$

where  $r', \theta'$  are polar coordinates and  $H_m$  denotes the Hankel functions of the first kind. The coefficients  $a_m(\omega')$  are obtained in the comoving frame of reference and depend on the direction of incidence  $\underline{k}'$  (suppressed in (27)) and the frequency of excitation  $\omega'$ . By using the Sommerfeld plane wave representation of the Hankel functions {Stratton, 1941}, (27) is recast as

$$\begin{aligned} u'(r', t') &= \frac{f'}{\pi} \int_{\theta' - \pi/2 + i\infty}^{\theta' + \pi/2 - i\infty} e^{ik' r' \cos(\theta' - \tau') - i\omega' t'} g'(\tau') d\tau' \end{aligned} \tag{28}$$

where the scattering amplitude is defined as

$$g'(\theta') = \sum_{m=-\infty}^{\infty} a_m e^{im\theta'} \tag{29}$$

and continued in (28) into the complex plane  $\tau'$ . Applying the inverse transformation of (25) for the present geometry we obtain

$$u(\underline{r}', t') = \frac{\gamma f'}{\pi} \int_{\theta' - \pi/2 + i\infty}^{\theta' + \pi/2 - i\infty} e^{ik'r' \cos(\theta' - \tau') - i\omega't'} g(\tau') d\tau'$$

$$g(\tau') = (1 + \beta \cos \tau') g'(\tau') \quad (30)$$

where  $u(\underline{r}', t')$  is the field measured in the laboratory frame, but still expressed in terms of the comoving variables  $\underline{r}', t'$  for convenience. By substitution of the Lorentz transformation one finds  $u(\underline{r}'(r, t), t'(r, t))$ , now in terms of the laboratory coordinates  $\underline{r}, t$ . However, this last step leads to cumbersome expressions which are not easily manipulated. Before applying the Lorentz transformation, it is convenient to use the transition from (28) to (27) and recast (30) as

$$u(\underline{r}', t') = \gamma f' \sum_{m=-\infty}^{\infty} i^m b_m(\omega') H_m(k'r') e^{im\theta' - i\omega't'}$$

$$b_m = a_m + (a_{m-1} + a_{m+1})\beta/2 \quad (31)$$

Note that to the first order in  $\beta$ , in (31)  $a_{m-1}, a_{m+1}$  are evaluated at the laboratory frequency  $\omega$  of the incident wave. However,  $a_m(\omega')$  contain first order in  $\beta$  terms, and if we wish to use  $\omega$  in (31), we must expand

$$a_m(\omega') = (1 + \beta \frac{\partial}{\partial \beta}) a_m |_{\omega' = \omega} \quad (32)$$

Similar expansions can be applied to other factors in (31) containing  $\omega', k'$ . The structure of (31) suggests, in a sense, that velocity-induced multipole terms appear, which are of order  $\beta$  [Censor, 1972a]. The corresponding three dimensional problem has been considered too [Censor, 1972a], the details are much more complicated.

The conclusion of this subsection is that the Einstein recipe is indeed feasible and adequate for problems involving uniform rectilinear motion.

Multiple scattering. Scattering by a collection of objects, moving in general

with different velocities, conforms to Einstein's recipe only to the extent that a successive scattering scheme is employed. Thus at each step Einstein's recipe is applied to one object, and all the pertinent waves are transformed into the comoving system of this object. In contradistinction, in velocity independent configurations a general multiple scattering theory has been developed [Twersky, 1962a, b, 1964, 1967; Burke et al., 1965]. The velocity-dependent multiple scattering problem is very complicated and the available literature is meager [Censor, 1970, 1972b; Pogorzelski, 1970]. The problem of velocity-dependent successive scattering is cumbersome and requires machine computation [Censor, 1972b]. A general theory, comparable to the velocity-independent class of problems, is to date nonexistent.

The feasibility of laboratory solutions. For the one-dimensional class of problems, solutions have been obtained either by means of the Einstein recipe or by a direct application of the boundary conditions in the laboratory system. In the latter case, the question of the laboratory boundary condition is to be answered. If special relativity is to be avoided (as was the case before the advent of the theory), simple energy consideration could be applied to some one-dimensional and plane interface oblique reference problems. The feasibility of the Einstein recipe for two- and three-dimensional problems has been amply demonstrated, above and in the cited literature. There exists also some literature [Cooper and Strauss, 1976, 1979, 1980; Cooper, 1979] contributing to the understanding of the problem in general, which is of a more mathematical nature. However, there are some fundamental difficulties restricting the use of direct laboratory solutions to the one-dimensional case.

The solvability of the one dimensional problem stems from the existence of the D'Alembert solution for the wave equation, as explained above. For uniform rectilinear motion, as shown above, the solution can be extended to the oblique incidence case. For two- and three-dimensional problems we do not have to

date even one analytical solution obtained in the laboratory system. For these cases we do not have a D'Alembert solution of the wave equation either. This seems to be more than a mere coincidence. Can we hope to find exact analytical solutions to two- and three-dimensional velocity dependent scattering problems without using space-time transformations at all? The conjecture of the present author is that such solutions are impossible. In the present state of the art, even producing one such solution would be a great contribution. For example, take the case of scattering by a cylinder, for which we have solution (31), relativistically exact, based on the Einstein recipe. Could we arrive at this solution, for the perfectly conducting circular cylinder, say, without using any transformations based on Einstein's recipe? Observe, for example, that even the frequency transformation (26) should appear automatically in a laboratory-based solution. This seems to be unattainable at this time. It seems safe to say, although it cannot be proved, that two- and three-dimensional laboratory based solutions are impossible because of the dispersive nature of such problems.

#### TWO AND THREE DIMENSIONS: NONUNIFORM MOTION

For nonuniform motion, the boundary conditions used to date in all available studies are based on the instantaneous velocity assumption, discussed above. Since this is at best an approximation (and at worst part of the "fiction" associated with our present subject), it can be said that there exists not even a single exact solution to a nonuniform velocity dependent scattering problem. Loosely speaking, if the relative rate of change of velocity is very small during the time of one period of the wave, i.e.,  $(dv/dt)/v \ll 1/T$ , where  $T$  is the time of a period, then the "approximation" might be plausible. But nevertheless, this method is used much more liberally, by many authors (including

the present one). The assumption of instantaneous velocity has also been applied to rotating systems {Van Bladel, 1976}, where it might be even better justified, because in the case of rotating circular cylinders and spheres the problem does not give rise to Doppler frequency shifts. However, this class of problems is outside of the scope of the present theme.

Unlike previous sections, it is difficult to classify the present problem into subsections. We wish to mention some work available on expanding cylinders and spheres, involving various simplifying assumptions {Lam, 1968; Censor, 1972c, 1973b; Pogorzelski, 1973}. No comparative studies have been attempted, and the results do not indicate a pattern conducive to a general method.

Apart from the part of the problem involving boundary conditions, there is the question of adjusting the temporal and spatial behavior of the scattered wave, such that whatever the chosen boundary conditions, they will be satisfied at the surface of the scatterer for all  $t$ . This problem has been satisfactorily solved for the one dimensional case by the formalism described above (16) - (20) {Censor, 1973a}. The problem of periodically oscillating objects has also been discussed {Cassedy, 1965; Censor, 1972a, 1973a; Borkar and Yang, 1975; Cooper and Strauss, 1981}. Especially noteworthy is a recent study by De Zutter {1982}, who deals with the problem of oblique incidence of a plane harmonic wave, reflected by a plane mirror. By employing a spectral analysis of the incident and reflected waves at the surface of the mirror, a solution is derived in the form of an infinite series of partial waves. De Zutter succeeds in taking into account the angular spectrum of the scattered wave, describing the effect of the nonuniform velocity on the aberration. Interestingly, his solution predicts the existence of evanescent waves. The only critical comment that can be made about this

result, as above, is that the question of using instantaneous velocities in boundary conditions is open to further argument.

In the context of acoustics [Censor, 1972d], it has been shown that periodically perturbed surfaces give rise to similar spectra, i.e., sidebands  $\omega \pm n\Omega$ ,  $n = 1, 2, \dots$ , where  $\Omega$  is the mechanical frequency of oscillation. In view of the Floquet theory [Ince, 1956; Friedman, 1956] for differential equations with periodically changing parameters, this fact is not surprising.

QUO VADIS?

Almost one and a half century after Doppler's original work [1842] and close to a century after Einstein's [1905] contribution, the lack of a comprehensive physical and mathematical theory (except for the simplest cases) for the Doppler effect is besetting engineers, physicists and mathematicians. It forces us to use "brute force," non self-consistent methods. Until more theoretical aspects are clarified, research should attempt at evaluating existing methods. This can be done, to some extent, by means of computer generated solutions.

GTD approach. Consider arbitrarily moving impenetrable (i.e., perfectly conducting) surfaces. If we deal with high frequencies, in the sense that geometrical optics is adequate, and also with velocity fields that change negligibly during the period of the wave, then it seems plausible to use (21), assuming the validity of the instantaneous velocity method. This, of course, is again what we provocatively termed "fiction," but it can be checked against the relativistically exact solutions for known problems, e.g., moving cylinder and sphere. This cannot provide answers as to the validity of the instantaneous velocity boundary conditions, which has to be solved theoretically. Interesting problems can be solved for complicated surfaces whose different parts move with different velocities. In general, the surface of a fixed object can be descri-

bed by  $F(\underline{r}) = 0$ . If the motion depends on time, the surface will be described by

$$F(\underline{r}, t) = 0 \tag{33}$$

The velocity normal to the surface, which causes the Doppler effect, is found in the following manner. Representing (33) as  $t = T(\underline{r})$  and expanding about a certain location  $\underline{r}_0$ , we get

$$t = t_0 + \underline{s} \cdot (\underline{r} - \underline{r}_0) + \dots$$

$$\underline{v} = \underline{v}(\underline{r}_0) \quad \underline{s} = \left. \frac{\partial t}{\partial \underline{r}} \right|_{\underline{r} = \underline{r}_0} \tag{34}$$

Provided higher terms of the expansion are negligible (which does not happen for periodic motion, for example), one may use the first two terms in (34) and exploit  $\underline{s}$ , the slowness function (so called because of the dimensions, which are inverse to velocity), i.e.,

$$dt = \underline{s} \cdot d\underline{r} \tag{35}$$

Freezing the time, i.e., taking  $dt = 0$ , we deal with the configuration of the surface at a certain time, say  $t = t_0$ . Hence  $d\underline{r}$  is an element of this surface and (35) shows that the slowness vector is perpendicular to this surface. The normal velocity  $v$  of the surface element is given by

$$v = \frac{1}{|\underline{s}|} \hat{\underline{s}} \tag{36}$$

which is entered in (21), to solve for the ray or rays bounced off the moving surface. This argument has been given to make the point that unless the truncation of (34) can be justified, we do not have ground for using the instantaneous velocity method.

Low frequencies. When dealing with objects whose dimensions and curvature do not admit geometrical optics considerations, we need a different approach. It is suggested that the surface be replaced by a collection of small objects (cylinders or spheres, say), moving with the local instantaneous velocity, and a velocity-dependent multiple scattering

scheme be used. To date it is clear how to deal with the individual cylinder or sphere in the near field {Censor, 1972a}; also, the mechanism of velocity-dependent successive scattering is clear {Censor, 1972b}. The detailed problem is very complicated, and only a computational program can yield results.

To date, the schemes proposed in this section have not been carried out.

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