

Energy balance and radiation forces for arbitrary moving objects

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In scattering processes involving moving objects an exchange of electromagnetic and mechanical power takes place. The problem is considered for arbitrary objects in one, two, and three dimensions. It is shown that the energy balance depends on the scattering amplitude, evaluated at the frequency of excitation, on this frequency itself, and on the velocity in question. Three cases are considered: objects moving in free space, objects moving together with the surrounding medium, and objects moving through a medium, at rest with respect to the observer. An interesting result is the fact that objects moving in the direction of the incident wave cause a loss of electromagnetic energy smaller than the gain occurring when the objects move in the opposite direction. This is a first-order effect in the velocity.

INTRODUCTION

An early history of the problem of radiation pressure is given by *Debye* [1909], who also cites earlier original studies, e.g., Maxwell. The problem of the radiation force on a moving mirror played an important role in the early days of the theory of relativity; *Pauli* [1958] gives a comprehensive survey of this phase. The problem is treated by *Einstein* [1905]; this treatment marks the beginning of a consistent formalism based on the Lorentz transformation and the invariance of Maxwell's equations. Recently *Res-trick* [1968] considered the problem of energy balance in the case of a perfectly conducting moving sphere. His results hinge on Debye's work. The motivation for Debye's paper was a problem in the physics of comets. Similar astrophysical problems are being re-examined these days, and the significance of radiation pressure is being considered in view of hitherto unknown phenomena, such as the solar wind. From an engineering point of view, the effect of radiation pressure on the trajectory of space vehicles must be evaluated. Hence the problem has lost none of its vigor over the years.

In many cases, it is the total radiation force acting on a body that is of interest, rather than the radiation pressure at each point on the surface. It is shown subsequently that the radiation force acting on an object depends on the scattering amplitude g , a function of directions only, related to the far-zone field. For many practical cases this function is not

known analytically, but it may be easily measured in a scattering experiment conducted in the far field. In the latter case the relation of the scattering amplitude to the detailed geometry and the constitutive parameters is of no importance.

The one-, two-, and three-dimensional problems for objects moving in free space are considered. This is a special case of the class of problems involving objects moving together with the surrounding medium. The third class of problems considered involves objects moving through a medium at rest. To date this problem can be treated exactly only for the one-dimensional case; for the more interesting problems in two and three dimensions only a first-order formalism is available [*Censor*, 1970].

In the case of moving objects the energy balance depends on the direction of the velocity relative to the direction of propagation of the incident wave. For the one-dimensional case it is shown that if an object is illuminated by two waves of equal intensity coming from opposite directions, it tends to slow down.

STATEMENT OF THE PROBLEM

The geometry of the problem is defined in Figure 1. In frame of reference Γ a surface S is at rest. For a certain period of time a moving object is contained inside this surface. The rest frame attached to the object is Γ' . Observed from Γ , the origin of Γ' is seen to move at a constant velocity $\mathbf{v} = v\hat{x}$.

For a given excitation wave we are interested in the net flux of electromagnetic power through the

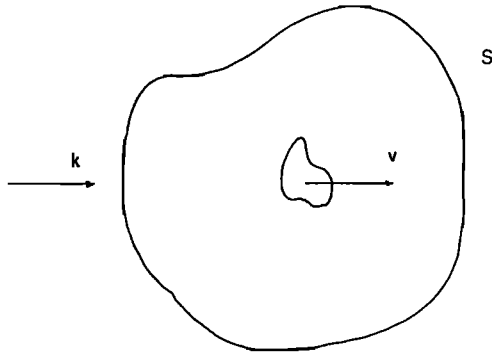


Fig. 1. In system of reference Γ a surface S is at rest. The scatterer is attached to system of reference Γ' , moving with velocity v as observed from Γ . The incident wave propagates in direction $\hat{k} = \hat{v}$.

surface S . In the case of objects at rest, if the internal region is lossless, the net flux is zero. In the present case the radiation forces acting on the moving objects dissipate power; hence electromagnetic energy is gained or lost. Analytically, we have to make an appropriate statement regarding the Poynting theorem, which follows directly from Maxwell's equations for the electromagnetic fields. In two and three dimensions, even though the incident plane wave is taken as time-harmonic, the scattered wave depends on time in a more complicated way. Therefore the Poynting theorem must be considered with respect to arbitrary time-dependent complex fields as discussed, for instance, by *Stratton* [1941]. For sake of symmetry both the real and imaginary parts are included here, and the time-averaged Poynting theorem takes the form

$$\begin{aligned}
 (1/2) \oint_S [\langle \text{Re}(\mathbf{E}) \times \text{Re}(\mathbf{H}) \rangle & \\
 + \langle \text{Im}(\mathbf{E}) \times \text{Im}(\mathbf{H}) \rangle] \cdot d\mathbf{S} & \\
 = (1/4) \oint_S \langle \langle \mathbf{E} \times \mathbf{H}^* \rangle + \langle \mathbf{E}^* \times \mathbf{H} \rangle \rangle \cdot d\mathbf{S} & \\
 = -(1/4) \int_V [\langle \mathbf{E} \cdot \mathbf{j}^* \rangle + \langle \mathbf{E}^* \cdot \mathbf{j} \rangle + \langle \mathbf{H} \cdot \partial_t \mathbf{B}^* \rangle & \\
 + \langle \mathbf{H}^* \cdot \partial_t \mathbf{B} \rangle + \langle \mathbf{E} \cdot \partial_t \mathbf{D}^* \rangle + \langle \mathbf{E}^* \cdot \partial_t \mathbf{D} \rangle] dV & \quad (1)
 \end{aligned}$$

where Re , Im signify the real and imaginary parts, respectively, the asterisk denotes the complex conjugate, and ∂_t stands for the partial time derivative. The angle brackets indicate time averaging according to

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T dt \quad (2)$$

What is meant by (2) is that the time T involves many cycles of the waves, while the object stays within S . For (2) to be rigorous S must extend to infinity in the direction of motion and $T \rightarrow \infty$. For time-harmonic functions, when one deals with the case $v = 0$, or plane waves as they occur in the one-dimensional case, (1) coincides with the well-known expression for the flux $(\frac{1}{2}) \text{Re} \mathbf{E} \times \mathbf{H}^*$. The interpretation of the volume integral in (1) for moving media caused some controversy in the literature [*Compton and Tai*, 1964; *Berger and Griemsmann*, 1967]. Here it is understood that the volume integral in (1) involving \mathbf{j} , \mathbf{j}^* describes the time-averaged power lost or gained by the electromagnetic field. Even if the media are not dissipative, charges are induced on the objects by the excitation wave; and since the objects are in motion, currents are created. Hence this term describes the power dissipated or gained when the radiation forces act on the moving objects. In the case of lossy media it takes into account Joule losses as well. Since constant velocities are assumed, other mechanisms of energy conversion are excluded, e.g., recoil energy of the objects under the impact of incident radiation. The remaining terms in (1), involving ∂_t , are considered to be the rate of change of stored energy. The problem of defining an energy-density function is not crucial for the subsequent considerations. For simple moving media Minkowski's constitutive relations [*Sommerfeld*, 1952] can be written in the form

$$\begin{aligned}
 \mathbf{D} &= \epsilon \boldsymbol{\alpha} \cdot \mathbf{E} - \boldsymbol{\beta} \cdot \mathbf{H} \\
 \mathbf{B} &= \mu \boldsymbol{\alpha} \cdot \mathbf{H} + \boldsymbol{\beta} \cdot \mathbf{E} \\
 \boldsymbol{\alpha} &= (1 - v^2/C^2)^{-1} [(c^{-2} - C^{-2})\mathbf{v}\mathbf{v} + \mathbf{I}/\gamma^2] \\
 \boldsymbol{\beta} &= (1 - v^2/C^2)^{-1} (c^{-2} - C^{-2})\mathbf{v} \times \mathbf{I} \\
 \gamma &= (1 - \beta^2)^{-1/2} \\
 \beta &= v/c
 \end{aligned} \quad (3)$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are dyadics, \mathbf{I} is the idemfactor dyadic, and C and c are the phase velocity in the refractive medium at hand and in free space respectively. For free space, $\boldsymbol{\beta} = 0$, $\boldsymbol{\alpha} = \mathbf{I}$, and ϵ , μ are taken as the constitutive coefficients ϵ_0 , μ_0 respectively, pertinent to free space. By substituting (3) into (1) and exploiting the properties of the antisymmetric dyadic $\boldsymbol{\beta}$ one gets

$$(1/8) \int_V \partial_t [\langle \mathbf{H}^* \cdot \mathbf{B} \rangle + \langle \mathbf{H} \cdot \mathbf{B}^* \rangle + \langle \mathbf{E}^* \cdot \mathbf{D} \rangle + \langle \mathbf{E} \cdot \mathbf{D}^* \rangle] dV \quad (4)$$

for the rate of change of stored energy.

The incident wave is taken as a transversal time- and space-harmonic plane wave in the frame of reference of the medium at rest. According to the transformation formulas for plane waves, (e.g., *Censor* [1969, 1970]), this implies that in Γ' of the object at rest we are dealing with time-harmonic waves involving, for example, $e^{-i\omega't'}$. The complex conjugate factors in (4) involve

$$x' = \gamma(x - vt) \quad (5)$$

according to the Lorentz transformation pertinent to the geometry of Figure 1. It follows that

$$\begin{aligned} \partial_t f(x - vt, y, z) \\ = -v \partial_x f = -\mathbf{v} \cdot \nabla f = -\nabla \cdot \mathbf{v} f \end{aligned} \quad (6)$$

where f is the expression in brackets in (4). By means of the Gauss theorem (4) can be transformed into a surface integral. Physically this means that the only mechanism by which the stored energy is changed is by convection across the surface S . The net power flowing out of S is not zero, and it constitutes the power dissipated by a force \mathbf{F} , acting on the moving object in direction \mathbf{v} ,

$$\begin{aligned} \oint_S [(\mathbf{E} \times \mathbf{H}^*) + (\mathbf{E}^* \times \mathbf{H}) - (1/2)\langle \mathbf{H}^* \cdot \mathbf{B} \rangle \\ + \langle \mathbf{H} \cdot \mathbf{B}^* \rangle + \langle \mathbf{E}^* \cdot \mathbf{D} \rangle + \langle \mathbf{E} \cdot \mathbf{D}^* \rangle] \mathbf{v} \cdot d\mathbf{S} \\ = - \int_V (\langle \mathbf{E} \cdot \mathbf{j}^* \rangle + \langle \mathbf{E}^* \cdot \mathbf{j} \rangle) dV \\ = -4\mathbf{F} \cdot \mathbf{v} = -4F_v \beta c \end{aligned} \quad (7)$$

Since a surface integral is to be evaluated, the problem finally involves far-field forms, i.e., F_v depends on the scattering amplitude for the object at hand. For $v = 0$ the radiation force acting on the object at rest is obtained, as shown subsequently. For a lossy scatterer the volume integral in (7) includes the Joule losses in the scatterer; for brevity, only lossless objects are considered. The computation of (7) for various cases is the subject of the subsequent sections.

THE PROBLEM OF THE SLAB

The problem of radiation pressure acting on a moving mirror was considered by *Einstein* [1905],

who compared the energy flux entering and leaving the surface of the moving mirror and attributed the difference to the power dissipated by the radiation forces. This approach is entirely different from the present one, depicted in Figure 1. It is not clear how Einstein's method may be used for two- and three-dimensional problems without actually finding the fields at the surface of the moving object in question. Since the one-dimensional case serves mainly to compare the methods, for simplicity the discussion is limited to normal incidence.

In free space the incident wave is specified in Γ by

$$\mathcal{y} A e^{ikz - i\omega t} \quad (8)$$

where \mathcal{y} is the direction of polarization of the \mathbf{E} field, say, therefore the \mathbf{H} vector points in the \hat{z} direction. In Γ' one observes that

$$\begin{aligned} \mathcal{y} A' e^{ik'x' - i\omega't'} \\ A'/A = \omega'/\omega = k'/k \\ = [(1 - \beta)/(1 + \beta)]^{1/2} \equiv B^{1/2} \end{aligned} \quad (9)$$

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - vx/c^2)$$

A slab is given in Γ' , characterized by the constitutive coefficients μ_i , ϵ_i and the width. For a lossless slab the transmission and reflection coefficients, $T(\omega')$ and $R(\omega')$ respectively, satisfy

$$|R|^2 + |T|^2 = 1 \quad (10)$$

evaluated at the excitation frequency ω' ; this is a statement of the conservation of energy in Γ' . In Γ' the transmitted wave equals the incident wave times $T(\omega')$, the reflected wave has $-ik'x'$ in the phase, and the amplitude contains $R(\omega')$ as a factor. Transforming back into Γ yields

$$\begin{aligned} \mathcal{y} A T(\omega') e^{ikz - i\omega t} \\ \mathcal{y} B A R(\omega') e^{-iB(kz + \omega t)} \end{aligned} \quad (11)$$

for the transmitted and reflected waves, respectively. The corresponding magnetic fields point in the \hat{z} , \hat{z} , $-\hat{z}$ directions, respectively, and all have $1/Z$ as a factor, where $Z = (\mu_e/\epsilon_e)^{1/2}$, index e referring to the external region, free space in the present case.

The evaluation of the surface integral (7), in conjunction with (10) yields

$$A^2 R^2 \beta (1 - \beta) / [Z(1 + \beta)] = F_v \beta c \quad (12)$$

The radiation force is therefore

$$F_s = \epsilon_0 A^2 R^2 (1 - \beta)/(1 + \beta) \quad (13)$$

which agrees with *Einstein's* [1905] result for $|R| = 1$. For $\beta = 0$ the force on a slab at rest is obtained. Consequently in Γ' the radiation pressure on a perfect reflector $R^2 = 1$ is $\epsilon_0 A^2$. This is consistent with the result obtained by application of the Maxwell stress tensor (e.g., *Panofsky and Phillips* [1955], chapter 6). Thus the rms electrical energy density of the incident wave is $\epsilon_0 A'^2/4$; it should be doubled, since half the energy is stored in the magnetic field, and doubled once more to take into account the reflected wave (in the present case superposition of energies in the above manner is permissible). Therefore the pressure is equal to the average energy density. Furthermore, $\epsilon_0 A'^2$ is equal to (13), in view of the transformation formula for A , (9). This is a verification of the invariance of the pressure [*Pauli*, 1958, pp. 97, 133].

Consider now a slab illuminated by two waves of equal amplitude, incident from opposite directions. For simplicity let the frequencies be slightly different, such that in Γ' the excitation frequency will be identical. By inspection of (13), the total force acting on the slab will now be seen to be

$$\begin{aligned} F \cdot \hat{\beta} &= \epsilon_0 A^2 R^2 [(1 - \beta)/(1 + \beta) - (1 + \beta)/(1 - \beta)] \\ &= -\epsilon_0 (2AR\gamma)^2 \beta \end{aligned} \quad (14)$$

The result indicates a first-order effect in the velocity; the negative sign means that there is a gain of electromagnetic energy, pumped into the system. This result is unaffected by change of direction of the velocity, if the slight difference in frequencies of excitation is negligible. Equation 14 suggests that if the slab is of finite mass and is left to its own, it will be slowed down and finally come to rest in the frame of reference where the two waves possess the same amplitude, or, more rigorously, in the frame of reference where AR is the same for both waves. This takes care of the different excitation frequencies too. This 'radiation friction' phenomenon raises an interesting speculation concerning interstellar matter. It suggests that matter far away from strong sources of radiation, if it is more or less evenly illuminated by the distant stars, will tend to slow down in the absence of other mechanisms that might affect its motion.

Now consider a slab moving together with the surrounding medium. The slab and the medium are at

rest in Γ' ; the observer and S are at rest in Γ , according to Figure 1. For the present case of a wave propagating in the direction of motion of the medium, (8) specifies the incident wave, with a suitable propagation constant k . For this case the wave is transversal both in Γ and in Γ' . The general transformation formula for plane waves in simple refractive media has been given (e.g., by *Censor* [1969]):

$$\begin{aligned} \mathbf{A}_1 &= \{[(1 - \gamma)\hat{e}_0 + \gamma v_0 \hat{k}_0 / C_0] \hat{e}_0 \\ &\quad + \gamma(1 - \hat{v}_0 \cdot \hat{k}_0 / C_0) \mathbf{1}\} \cdot \mathbf{A}_0 \end{aligned} \quad (15)$$

applicable to $\mathbf{A} = \mathbf{E}, \mathbf{H}$ fields, where index 0 refers to Γ_0 the frame of reference of the medium at rest; observed from Γ_0 , frame Γ_0' is seen to move in direction v_0 . For the present case this prescribes the following transformation for the amplitude:

$$A' = A / [\gamma(1 + v/C)] \quad C = (\mu_s \epsilon_s)^{-1/2} \quad (16)$$

where C is the phase velocity in the medium at rest. Cancelling A' , for the reflected wave one gets

$$A_r / A = (1 - v/C) / (1 + v/C) \quad (17)$$

For the present case the same transformation applies to frequencies as well, but not to the propagation constants. In order to evaluate (7) one needs the constitutive relations. Here [*Censor*, 1969] Minkowski's constitutive relations may be replaced by the scalar parameters

$$\begin{Bmatrix} \epsilon_s \\ \mu_s \end{Bmatrix} = \begin{Bmatrix} \epsilon_0 \\ \mu_0 \end{Bmatrix} \frac{1 + vC/c^2}{1 + v/C} \quad (18)$$

for the incident and transmitted waves in Γ ; for the reflected wave a minus sign appears in the numerator and denominator of (18). By means of (7) we obtain

$$F_s = \epsilon_s A^2 R^2 / [\gamma(1 + v/C)]^2 \quad (19)$$

for $C = c$ this yields (13) again, for $v = 0$ we get the expected force in Γ' , equal to the energy density. By inspection of (16) one establishes that F_s is again invariant.

The last case to be considered in one dimension concerns a slab moving through a medium without perturbing it. The slab might be an artificial dielectric consisting of a material mesh through which the medium can pass freely. The problem might also bear on the case of scattering by a shock wave propagating in a medium and changing locally the constitutive parameters. The problem of scattering by a slab containing a medium that flows perpendicularly

with respect to the interfaces has been discussed by *Censor* [1969]. Similarly, by considering the multiple scattering processes at the interfaces, new reflection and transmission coefficients may be defined for the present problem. In Γ we now have the external medium at rest. The incident plane wave is specified by (8). The transformation into Γ' of the slab at rest is given, similarly to (16), by

$$A' = A\gamma(1 - v/C) \quad (20)$$

The amplitude of the reflected wave obeys (17). The constitutive relations are simply $\mathbf{D} = \epsilon_s \mathbf{E}$, $\mathbf{B} = \mu_s \mathbf{H}$. Evaluation of (7) yields

$$F_s = \epsilon_s A^2 R^2 (1 - v/C)/(1 + v/C) \quad (21)$$

In order to find the force in Γ' it is not sufficient to put $v = 0$, as in the former cases, because in Γ' the slab is at rest but the external medium is in motion. Using (18), (20), and (10), which is valid too [*Censor*, 1969], and computing the pressures from the energy density according to $(\frac{1}{2})(\mathbf{D} \cdot \mathbf{E}' + \mathbf{B}' \cdot \mathbf{H}')$, yields the same result (21); hence the invariance of the radiation pressure has been verified once more.

THE PROBLEM OF THE CYLINDER

The incident wave is given in Γ (\mathbf{E} field, say, polarized along the z axis of the cylinder) by (8). The scattered field in Γ' is

$$\begin{aligned} u'(\mathbf{r}', t') &= A' \sum_{m=-\infty}^{\infty} i^m a_m H_m(k'r') e^{im\theta' - i\omega't'} \\ &= A' \int_{c_s} e^{ik'r' \cos(\theta' - \tau') - i\omega't'} g(\tau') d\tau'/\pi \\ g(\theta') &= \sum_{m=-\infty}^{\infty} a_m e^{im\theta'} \end{aligned} \quad (22)$$

where $H_m = H_m^{(1)}$ stands for the Hankel function of the first kind of order m , c_s is an appropriate Sommerfeld contour in the complex plane τ' . Using the relation

$$H_m^* = H_m^{(2)} = (-1)^m H_m(-k'r') \quad (23)$$

for integral number m , the complex conjugate of u' (22) can be written as

$$\begin{aligned} u'^* &= A' \sum_{m=-\infty}^{\infty} i^m a_m^* H_m(-k'r') e^{-im\theta' + i\omega't'} \\ &= A' \int_{c_s} e^{-ik'r' \cos(\theta' - \tau') + i\omega't'} g^*(\tau') d\tau'/\pi \end{aligned} \quad (24)$$

where A' , k' are considered real. It has been shown by *Censor* [1967] that the scattered field u measured

by an observer in Γ , but expressed in terms of \mathbf{r}' , t' coordinates, can be written as

$$\begin{aligned} u(\mathbf{r}', t') &= \gamma A' \sum_{m=-\infty}^{\infty} i^m b_m H_m(k'r') e^{im\theta' - i\omega't'} \\ b_m &= a_m + (a_{m+1} + a_{m-1})\beta/2 \end{aligned} \quad (25)$$

In the complex integral representation (22), (24) one multiplies the integrand by $\gamma(1 + \beta \cos \tau')$ to get u , u^* respectively. The \mathbf{H} fields associated with (22), (24) are obtained in Γ according to Maxwell's equations, by superposing the \mathbf{H} plane waves corresponding to the \mathbf{E} waves in the integrand. Mathematically, the operations on the integrand are justified by the rule of differentiation of an integral with respect to a parameter (note that the integrand vanishes at the limits of c_s). The field in Γ can be found by various methods, of which the following is an example. Since the phase is an invariant, it can be written as

$$\begin{aligned} k'r' \cos(\theta' - \tau') - \omega't' \\ = kr \cos(\theta - \tau) - \omega t = k_x x + k_y y - \omega t \end{aligned} \quad (26)$$

and the operations according to $\nabla \times \mathbf{E} = -\mu_s \partial_t \mathbf{H}$, in free space, yield the field directly in Γ . The result is then expressed in terms of τ' , the direction of propagation in Γ' , by substitution of the aberration formulas

$$\cos \tau = (\cos \tau' + \beta)/(1 + \beta \cos \tau') \quad (27)$$

$$\sin \tau = \sin \tau' / [\gamma(1 + \beta \cos \tau')]$$

This yields

$$\begin{aligned} H_u &= \frac{A'}{Z} \int_{c_s} e^{ik'r' \cos(\theta' - \tau') - i\omega't'} g(\tau') \\ &\quad \cdot [\hat{x} \sin \tau' - \hat{y} \gamma(\cos \tau' + \beta)] d\tau'/\pi \end{aligned}$$

and a similar expression for H^* , with g^* and the complex conjugate phase.

It is now necessary to form products, take time averages, and integrate on the surface according to (7). One is tempted to first take the far-field forms and work with these simple expressions. But in the far field $x \gg vt$ for the range of t of interest, and it will be shown that this introduces an ambiguity. When the near-field forms are manipulated first, and then the asymptotic approximations are made, it will become clear how the latter may be used directly, e.g., for three-dimensional cases.

For example, the terms EE^* will be considered in detail. Forming the product in terms of the complex integral representation and time-averaging the inte-

grand yields

$$\begin{aligned}
 \langle EE^* \rangle &= A^2 + \gamma^2 A'^2 \lim_{T \rightarrow \infty} \frac{1}{2T} \\
 &\cdot \iint \left\{ e^{i\gamma z(k'_{rz} - k'_{rz}) + i v(k'_{rv} - k'_{rv})} \right. \\
 &\cdot \frac{\sin [\gamma v T(k'_{rz} - k'_{rz})]}{\gamma v T(k'_{rz} - k'_{rz})} g(\tau') g^*(\nu') (1 + \beta \cos \tau') \\
 &\cdot (1 + \beta \cos \nu') d\tau' d\nu' / \pi^2 \left. \right\} + \gamma A A' \lim_{T \rightarrow \infty} \frac{1}{2T} \\
 &\cdot \left\{ \int e^{i\gamma z(k' - k'_{rz}) - i k'_{rv} \nu'} g^*(\nu') (1 + \beta \cos \nu') \right. \\
 &\cdot \frac{\sin [\gamma v T(k' - k'_{rz})]}{\gamma v T(k' - k'_{rz})} d\nu' / \pi \\
 &+ \int e^{-i\gamma z(k' - k'_{rz}) + i k'_{rv} \nu'} g(\tau') (1 + \beta \cos \tau') \\
 &\cdot \left. \frac{\sin [\gamma v T(k' - k'_{rz})]}{\gamma v T(k' - k'_{rz})} d\tau' / \pi \right\} \quad (28)
 \end{aligned}$$

where $k'_{rz} = k' \cos \tau'$, etc. The contours of integration in the τ' or ν' planes involve θ' , a function of t ; hence the time-averaging of the integrand presupposes that at the point of observation the angular velocity of the scatterer is negligible. This is justified a posteriori by taking the saddle-point approximation of the integrals. Alternatively one may effect a change of variable that frees the contours from θ' and introduces it in the integrand. By performing an integration by parts and noting that the second integral vanishes as $r' \sim \infty$, the former procedure is justified again. The surface integration is performed on a circle of large radius r_γ in the $x_\gamma y$ plane, $x_\gamma = \gamma x$ (this is an ellipse in the xy plane),

$$\begin{aligned}
 \int \langle EE^* \rangle \nabla \cdot dS \\
 &= r_\gamma A^2 v \int_0^{2\pi} \cos \theta_\gamma d\theta_\gamma + (\gamma^2 A'^2 2v / \pi k') \\
 &\cdot \int_0^{2\pi} |g(\theta_\gamma)|^2 (1 + \beta \cos \theta_\gamma)^2 \cos \theta_\gamma d\theta_\gamma \\
 &+ \gamma A A' v (2r_\gamma / \pi k')^{1/2} \lim_{T \rightarrow \infty} \left[e^{-ik' r_\gamma + i\pi/4} \right. \\
 &\cdot \int_0^{2\pi} e^{ik' r_\gamma \cos \theta_\gamma} \Lambda(T, \theta_\gamma) g^*(\theta_\gamma) d\theta_\gamma + e^{ik' r_\gamma - i\pi/4} \\
 &\cdot \left. \int_0^{2\pi} e^{-ik' r_\gamma \cos \theta_\gamma} \Lambda(T, \theta_\gamma) g(\theta_\gamma) d\theta_\gamma \right]
 \end{aligned}$$

$$\begin{aligned}
 \Lambda(T, \theta_\gamma) &= (1 + \beta \cos \theta_\gamma) \cos \theta_\gamma \\
 &\cdot \sin [k' \gamma v T (1 - \cos \theta_\gamma)] / [k' \gamma v T (1 - \cos \theta_\gamma)] \quad (29)
 \end{aligned}$$

The first integral on the right-hand side (29) vanishes; the second one is left in the present form, and since θ_γ is a dummy variable, it may be replaced by an arbitrary angle α . The third and fourth integrals are evaluated by the method of steepest descent. At the stationary point $\theta_\gamma = 0$ the integrand vanishes in view of $T \rightarrow \infty$; finally, these two integrals yield

$$(4 A^2 v / k') \operatorname{Re} g(0) \quad (30)$$

On the other hand, if one starts from EE^* in the far field,

$$\begin{aligned}
 EE^* &\sim A^2 + (\gamma^2 A'^2 2 / \pi k' r') |g(\theta')|^2 (1 + \beta \cos \theta')^2 \\
 &+ \gamma A A' (1 + \beta \cos \theta') (2 / \pi k' r')^{1/2} \\
 &\cdot [e^{ik' (x' - r')} + i\pi/4 g^*(\theta') + e^{-ik' (x' - r') - i\pi/4} g(\theta')] \quad (31)
 \end{aligned}$$

then (noticing that $x \gg vt$ for the range of t of interest), substitution of $r_\gamma, x_\gamma, \theta_\gamma$ for r', x', θ' yields the time-averaged function directly. In this case it is not clear why the stationary point $\theta_\gamma = \pi$ must be excluded. But now that we are in possession of both (29) and (31), for other cases it is concluded that the far field may be used, provided only that the direction of incidence is considered for the steepest-descent integrals of the type given in (29). The other terms of (7) are treated in a similar fashion; finally, one gets

$$\begin{aligned}
 A^2 (1 - \beta) \operatorname{Re} g(0) + \frac{A'^2}{2\pi} \int_0^{2\pi} |g(\alpha)|^2 (1 + \beta \cos \alpha) \\
 \cdot (\gamma \sin^2 \alpha + \cos^2 \alpha) d\alpha = -F_\beta c Z k' / 2 \quad (32)
 \end{aligned}$$

For $\beta = 0$ Twersky's [1962] formula for the velocity-independent case is obtained:

$$-\operatorname{Re} g(0) = \frac{1}{2\pi} \int_0^{2\pi} |g(\alpha)|^2 d\alpha \quad (33)$$

Substituting (33) back into (32) yields

$$\begin{aligned}
 F_\beta c k' Z / (2 A'^2) \\
 &= \int_0^{2\pi} |g(\alpha)|^2 \{1 + \beta - (1 + \beta \cos \alpha) \\
 &\cdot [\gamma - (\gamma - 1) \cos^2 \alpha]\} d\alpha / 2\pi \\
 &= \frac{1}{2\pi} \int_0^{2\pi} |g(\alpha)|^2 \beta (1 - \cos \alpha) d\alpha + O(\beta^2) \quad (34)
 \end{aligned}$$

From (34) it follows for $\beta = 0$ that

$$F_{,ck'}Z/(2A^2) = \frac{1}{2\pi} \int_0^{2\pi} |g(\alpha)|^2 (1 - \cos \alpha) d\alpha \quad (35)$$

which is the force acting on an object at rest. The last expression (35) may be computed directly from the energy density on a large circle, which at each point is equal to the normal pressure. Keeping second-order velocity effects in (34) yields the first order velocity effect on the force; for this case the expression in braces in (34) is replaced by

$$\beta(1 - \cos \alpha - \frac{1}{2}\beta \sin^2 \alpha) \quad (36)$$

Next consider the problem of a cylinder embedded in a medium and moving together with it. The incident wave is defined as before, with the transformation (16) for the amplitude. The \mathbf{E} field results from multiplying the integrand in (22) by $\gamma[1 + (v/C) \cos \tau']$; the \mathbf{H} field is obtained by substituting v/C for β in (27), and similarly for \mathbf{E}^* , \mathbf{H}^* . The transformation for \mathbf{D} , \mathbf{B} fields [Censor, 1969] is obtained by substituting C/c^2 for $1/C$ in (15). Accordingly the \mathbf{D} field is obtained by multiplying the integrand in (21) by $\epsilon_r \gamma[1 + (vC/c^2) \cos \tau']$; similarly β in (28) is replaced by vC/c^2 . The \mathbf{D} , \mathbf{B} fields of the incident wave in Γ are given by $\mathbf{D} = \epsilon_r \mathbf{A}$, $\mathbf{B} = \mu_r \mathbf{A}/Z$ according to (18). The surface integral (7) is now evaluated in the far field, taking $\theta_r = 0$ as the stationary point for the steepest-descent integrals. This yields the analog of (32)

$$\begin{aligned} & A^2 \operatorname{Re} g(0)/[\gamma^2(1 + v/C)] \\ & + \frac{A'^2}{2\pi} \int_0^{2\pi} |g(\alpha)|^2 \left(1 + \frac{v}{C} \cos \alpha\right) \\ & \cdot (\gamma \sin^2 \alpha + \cos^2 \alpha) d\alpha = -F_{,vZk'}/2 \end{aligned} \quad (37)$$

For $C = c$, (32) is obtained once more as a special case; for $v = 0$ we have (33). Similarly to (34), we have an expression of the same form, with c replaced by C in the left-hand term and β replaced by v/C where it appears explicitly. Therefore (35) and (36) follow.

The problem of an object moving in a medium without perturbing it is considered by Censor [1970]. By inspection of the results in the far field it follows that, to the first order in the velocity, (37) and the expressions following it are valid for the present case too. Of course, g has now a velocity dependence that is implicit in these equations. Indeed, the slab results (20) and (21) for these two

cases of material media are also identical to the first order in the velocity.

THREE DIMENSIONS

The results for three-dimensional cases follow in a straightforward way from the previous one- and two-dimensional considerations. It is only necessary to discuss the case of an object moving together with the surrounding medium. To the first order in β the result is then valid for objects moving through a medium at rest. By taking $C = c$ the exact result for objects moving in free space is obtained. The incident wave propagates, as before, in direction \hat{x} , which is taken as the polar axis; the \mathbf{E} field is polarized in the \hat{z} direction, say, referred to as $\phi = 0$. Analogously to (22), (24), the scattered field is represented as a complex integral and the transformation (15) applied to g in the integrand. This yields the field in Γ , which in the far field takes the form

$$\begin{aligned} \mathbf{u} & \sim (A'/ik'r')e^{ik'r'-i\omega t'} [(1 - \gamma)\hat{x}(\hat{x} \cdot \mathbf{g}) \\ & - \gamma(v/C)\hat{r}(\hat{x} \cdot \mathbf{g}) + \gamma\left(1 + \frac{v}{C} \cos \theta'\right)\mathbf{g}] \end{aligned} \quad (38)$$

$$\mathbf{g} = \mathbf{g}(\theta, \phi)$$

For \mathbf{D} , \mathbf{B} fields (15) is rewritten with vC/c^2 instead of v/C . The corresponding \mathbf{H}' field in Γ involves A'/Z instead of A' and \mathbf{g}_H instead of \mathbf{g} , where

$$\mathbf{g}_H = \hat{r} \times \mathbf{g} \quad (39)$$

Therefore in (38) one gets the far zone scattered field \mathbf{H} by replacing \mathbf{g} by \mathbf{g}_H/Z . According to (7) we obtain

$$\begin{aligned} & A'^2 8\pi \operatorname{Re} \hat{z} \cdot \mathbf{g}(\hat{x})/[\gamma^2(1 + vC)] + A'^2 \\ & \cdot \int d\Omega \left\{ (1 - \gamma)\gamma \left(2 + \frac{v}{C} \cos \alpha\right) (\mathbf{g}_z \mathbf{g}_z^* + \mathbf{g}_{Hz} \mathbf{g}_{Hz}^*) \right. \\ & \left. + 2\left(1 + \frac{v}{C} \cos \alpha\right) (\gamma^2 \sin^2 \alpha + \cos^2 \alpha) \mathbf{g} \cdot \mathbf{g}^* \right\} \\ & = -4F_{,vZk'} \end{aligned} \quad (40)$$

$$\int d\Omega = \int_0^{2\pi} \int_0^\pi \sin \alpha d\alpha d\phi$$

For $v = 0$ this yields Twersky's [1967] form analogous to (33),

$$-\operatorname{Re} \hat{z} \cdot \mathbf{g}(\hat{x}) = \frac{1}{4\pi} \int d\Omega \mathbf{g} \cdot \mathbf{g}^* \quad (41)$$

Substitution of (41) into (40) yields an expression appropriate for deriving the force. To the first order

in the velocity,

$$2F, Zk'^2 C = A^2 \int d\Omega (1 - \cos \alpha) g \cdot g^* \quad (42)$$

which is the analog of (35).

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