Novel perspectives on low frequency scattering

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ABSTRACT

The Stevenson approach to low-frequency time-harmonic wave scattering, that expanded
the electric and magnetic fields in power series of $k$, essentially the inverse wavelength, is
scrutinized. Stevenson’s power series approach perforce implies a variable frequency $\omega$, i.e.,
a variable wave-number $k$, an assumption challenged here. Presently the three major linear
wave physics models: acoustics, electromagnetics, and elastodynamics, are put on an equal
footing by introducing the self-consistent system concept. Accordingly any low-frequency
series expansion starts with the pertinent Helmholtz equation. Far-field surface-integrals are
derived for each case. To verify our approach, an example of low-frequency electromagnetic
scattering by a long cylinder is elaborated, the results are compared to, and agree with the
exact Hankel-Fourier series solution.

1. INTRODUCTION AND OVERVIEW OF THE OLD THEORY

Low-Frequency wave theory is important in physics, used for example in the Rayleigh scatter-
ing theory that explains the blue color of the sky in daytime, and the red color at sunset. It was
therefore recognized that a concise mathematical theory is very desirable. One mathematical
advantage of the formalisms discussed below is that they replace solutions of Helmholtz wave
equations with solutions of corresponding (scalar and vector) Laplace equations, studied in
potential theory. Moreover, the latter are separable in more curvilinear coordinate systems,
thus leading to canonical solutions which are not available for the Helmholtz equation.1

As a prototype of low-frequency theories consider the analysis of Morse and Feshbach
(see,1 p. 1085), who represent the solution of the Helmholtz equation in terms of a series in
ascending powers of the constant $ik = i\omega/c$ . This constant may assume different values for
different problems, i.e., be a parameter in the context of a family of solutions. Unfortunately
they use the term ”power series”, but evidently this is not a series in ascending powers of a variable. We contend that the need to appreciate this distinction is crucial to the

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development of a consistent low-frequency theory. Accordingly we refer to the two kinds of series as constant-coefficient, and variable-coefficient series, respectively.

In 1953 Stevenson introduced a basic approach to low-frequency scattering that expanded the electric and magnetic fields in variable-coefficient series of the inverse wavelength \( k/2\pi = \omega/(2\pi c) \), and by comparing equal powers of \( k \), used these series to derive recurrence relations and distant field surface-integral representations. Many authors have subsequently used this method. A similar approach has been adopted in. The implicit assumption in Stevenson’s source paper is that these series are of the variable-coefficient type. We show here that this assumption is incorrect.

We also show that another ansatz inherent in the traditional theory, that of expanding both the \( E \) and \( H \) fields as independent series leads to inconsistencies. The correct procedure is to expand just one field (\( E \) say), and then to obtain the expansion of the other field (\( H \)) via the appropriate Maxwell curl relation. Only if this procedure is followed is a solution of Maxwell’s equations, or an equivalent self-consistent system of equations, guaranteed. A similar procedure applies to low frequency scattering in acoustics and elastodynamics.

Finally, the new theory is illustrated and verified by analyzing the problem of scattering from a perfectly conducting cylinder. We compare the new results obtained using our low-frequency algorithm with the well studied Hankel-Fourier series solution.

In order to contrast the present results with the Stevenson approach, and to introduce some useful notation and formulas, we present the derivation for the electromagnetic field, based on the legitimacy (which we contest) of using variable-coefficient series expansion in \( ik \).

We start with the source-free Maxwell equations in MKS units, (see e.g., Stratton), for time-harmonic fields with the time factor \( e^{-i\omega t} \), which is henceforth suppressed, in homogeneous, linear, isotropic media (e.g., free space):

\[
\begin{align*}
\partial_r \times E &= i\omega \mu H, \quad \partial_r \times H = -i\omega \varepsilon E, \quad \partial_r \cdot E = \partial_r \cdot H = 0 \\
E &= E(r), \quad H = H(r), \quad D = \varepsilon E, \quad B = \mu H
\end{align*}
\]

where in (1) \( \partial_r \) denotes the Nabla operator and \( \varepsilon, \mu \) are constant scalars. Stevenson’s theory starts with the assumption that the fields can be simultaneously expanded in variable-coefficient series in \( ik \):

\[
\begin{align*}
E\( r \) &= e_0 \sum_{n=0}^{\infty} (ik)^n E_n(r)/n! \\
H\( r \) &= h_0 \sum_{n=0}^{\infty} (ik)^n H_n(r)/n!
\end{align*}
\]

with constant amplitudes \( e_0, h_0 \), bearing the pertinent physical units.

Substituting (2) and (3) into (1) and equating powers of \( ik \) yields the recurrence relations

\[
\begin{align*}
\partial_r \times E_n &= h_0 c n \mu H_{n-1}, \quad h_0 \partial_r \times H_n = -\varepsilon e_0 c n E_{n-1} \\
\partial_r \cdot E_n(r) &= \partial_r \cdot H_n(r) = 0 \\
c &= (\mu \varepsilon)^{-1/2}, \quad E_n = E_n(r), \quad H_n = H_n(r)
\end{align*}
\]

Appropriate re-indexing has been done in the above equations to ensure that the summation range can be maintained as starting with \( n = 0 \). For cases where choosing \( e_0/h_0 = Z = \)
\((\mu/\varepsilon)^{1/2}\) is allowed, (4) simplifies to
\[
\begin{align*}
\partial_r \times \mathbf{E}_n &= n \mathbf{H}_{n-1}, \quad \partial_r \times \mathbf{H}_n = -n \mathbf{E}_{n-1} \\
\partial_r \cdot \mathbf{E}_n &= \partial_r \cdot \mathbf{H}_n = 0
\end{align*}
\]

The mathematical operations leading to (4) and (5) are only valid for variable-coefficient series where \(ik\) is a variable.

The results are far reaching, culminating in surface-integrals which can be found elsewhere.\(^5\)

Here is one way of seeing why (5) is inapplicable: Apply the curl operation to one of the equations in (5), note the div relations, and substitute the remaining curl relation. Repeat this for the other equation. We thus derive simultaneously
\[
\begin{align*}
\partial_r^2 \mathbf{E}_n(r) &= n(n-1) \mathbf{E}_{n-2}(r), \quad \partial_r^2 \mathbf{H}_n(r) = n(n-1) \mathbf{H}_{n-2}(r) \\
\partial_r \cdot \mathbf{E}_n(r) &= \partial_r \cdot \mathbf{H}_n(r) = 0, \quad \partial_r^2 = \partial_r \cdot \partial_r
\end{align*}
\]

Inasmuch that (5) provides the same number of equations as (6), we expect the two sets to be equivalent. However, it is well-known that reduced sets, obtained by substituting from the original equations, lose information and allow for solutions which do not satisfy the initial set. Solving the recurrence relations for \(\mathbf{E}_n\) and \(\mathbf{H}_n\) in (6) does not guarantee that (5) will be satisfied. The correct procedure is to solve for one of the fields, say \(\mathbf{E}_n\), and derive the associated \(\mathbf{H}_n\) from the appropriate Maxwell curl relation. Alternatively, first solve for \(\mathbf{H}_n\) and then derive the associated \(\mathbf{E}_n\). It follows that (2) and (3) cannot be stated simultaneously.

Notably for the treatment of acoustics, Dassios\(^5\) followed a different route. The fundamental equations for time-harmonic fields are
\[
\begin{align*}
\partial_r \cdot \mathbf{v}(r) - i\omega \gamma p(r) &= 0, \quad \partial_r p(r) - i\omega \rho \mathbf{v}(r) = 0
\end{align*}
\]

with \(p, \mathbf{v}, \gamma, \rho\) denoting acoustical pressure, velocity, compressibility, and mass-density, respectively.

Consistently with (2) and (3) one has to state
\[
\begin{align*}
p(r) &= q_0 \sum_{n=0}^{\infty} (ik)^n p_n(r)/n! \\
\mathbf{v}(r) &= v_0 \sum_{n=0}^{\infty} (ik)^n \mathbf{v}_n(r)/n!
\end{align*}
\]

with \(k = \omega/c = \omega(\gamma \rho)^{1/2}\), and obtain in a similar manner to (4)
\[
\begin{align*}
v_0 \partial_r \cdot \mathbf{v}_n(r) &= nc \gamma q_0 p_{n-1}(r), \quad q_0 \partial_r p_n(r) = n c \rho v_0 \mathbf{v}_{n-1}(r)
\end{align*}
\]

Similarly to (6), we derive from (10)
\[
\begin{align*}
\partial_r^2 p_n(r) &= n(n-1) p_{n-2}(r), \quad \partial_r^2 \mathbf{v}_n(r) = n(n-1) \mathbf{v}_{n-2}(r)
\end{align*}
\]

and all the reservations that applied to the set (6) follow for the set (11).

Interestingly, for acoustics the low-frequency analysis in\(^5\) starts from the Helmholtz equation on \((\partial_r^2 + k^2)p(r) = 0\) for the acoustic pressure \(p\), leading to the first recurrence relation
with the tacit assumption that for the field $v(\mathbf{r})$ one has to return to (7). This is the correct procedure as we understand it, following the prototype. Unfortunately the inconsistency of the two methods was not realized so far. Similar confusing arguments exist for elastodynamics too, with the unwarranted equating of powers of $ik$ in different series, leading to results that do not tally with the consistent systems discussed below.

2. CONSISTENT MAXWELL SYSTEMS

Starting again from (1), we derive by substitution the Helmholtz equations

\[
(\partial^2_r + k^2) \mathbf{E} = 0, \quad (\partial^2_r + k^2) \mathbf{H} = 0, \quad \partial_r \cdot \mathbf{E} = \partial_r \cdot \mathbf{H} = 0
\]  

(12)

Clearly the system (1) cannot be retrieved from (12). However, the set

\[
(\partial^2_r + k^2) \mathbf{E} = 0, \quad \partial_r \times \mathbf{E} = i\omega \mu \mathbf{H}, \quad \partial_r \cdot \mathbf{E} = \partial_r \cdot \mathbf{H} = 0
\]  

(13)

is necessary and sufficient for re-deriving the missing equation $\partial_r \times \mathbf{H} = -i\omega \varepsilon \mathbf{E}$. The sets (1) and (13) are therefore equivalent and we refer to (13) as the first consistent Maxwell system. The second consistent Maxwell system is given by

\[
(\partial^2_r + k^2) \mathbf{H} = 0, \quad \partial_r \times \mathbf{H} = -i\omega \varepsilon \mathbf{E}, \quad \partial_r \cdot \mathbf{E} = \partial_r \cdot \mathbf{H} = 0
\]  

(14)

from which $\partial_r \times \mathbf{E} = i\omega \mu \mathbf{H}$ is easily derived. The choice of one consistent system over another is a matter of convenience, when applying boundary conditions in a scattering problem, for example. It is easily verified that mixing consistent systems can result in over-determined systems in which a solution found by solving some of the equations might not be a solution to the remaining equations.

A similar situation exists for acoustics and elastodynamics, to which we refer later.

3. PLANE WAVES AND LOW FREQUENCY EXPANSIONS

Since the wave equation is linear, any solution can be constructed by combining basis solutions. The simplest example is a superposition (integral) of plane waves. However, in general such summations need to include wave solutions that are singular at the origin (e.g., spherical waves), that require complex $k$, even when studying lossless media characterized by real $k^2 = k \cdot k$. This is explained below in connection with the Sommerfeld-type integral representations of the special functions involved.

To illustrate these ideas in their simplest context, consider a plane electromagnetic wave at a single frequency, i.e., $k$ a constant vector, and its associated Taylor expansion. For simplicity choose $\mathbf{k} = k\hat{\mathbf{x}}$. A plane wave can serve as the incident wave for scattering problems

\[
E(\mathbf{r}) = \hat{\mathbf{e}}_0 e^{ik\mathbf{r}} = \hat{\mathbf{e}}_0 \sum_{n=0}^{\infty} (ik)^n (\hat{\mathbf{k}} \cdot \mathbf{r})^n / n!
\]

\[
E(x) = \hat{\mathbf{e}}_0 e^{ikx} = \hat{\mathbf{e}}_0 \sum_{n=0}^{\infty} (ik)^n (x)^n / n!, \quad k = k\hat{\mathbf{x}}
\]

\[
H(\mathbf{r}) = \hat{\mathbf{h}}_0 e^{ik\mathbf{r}} = \hat{\mathbf{h}}_0 \sum_{n=0}^{\infty} (ik)^n (\hat{\mathbf{k}} \cdot \mathbf{r})^n / n!
\]

\[
H(x) = \hat{\mathbf{h}}_0 e^{ikx} = \hat{\mathbf{h}}_0 \sum_{n=0}^{\infty} (ik)^n (x)^n / n!, \quad k = k\hat{\mathbf{x}}
\]  

(15)

polarized along the constant unit vectors $\hat{\mathbf{e}}, \hat{\mathbf{h}}$. 

In the language of complex variables the functions in (15), both in the exponential and
the series representations, are regular for small \( k \), i.e., no poles are present (see,\(^1\) p. 1085).
This is an essential test for the validity of representations similar to (15), discussed below.

It must be emphasized that a plane-wave (15), even for real wavenumber \( k \), can have
complex wavevector \( k \) if it propagates in complex directions (e.g., complex \( \hat{x} \)). Such waves
are called inhomogeneous plane waves (see,\(^7\) p. 360), and appear for example in the plane
wave integrals below. Consider a complex propagation vector

\[
\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I
\]

\[
k^2 = k \cdot k = \mathbf{k}_R \cdot \mathbf{k}_R - \mathbf{k}_I \cdot \mathbf{k}_I + 2i\mathbf{k}_R \cdot \mathbf{k}_I
\]

with subscripts \( R \) and \( I \) denoting real and imaginary components. Prescribing for complex
\( k \) that \( k^2 \), and therefore also \( k \), be real, implies that the real and imaginary components
must be perpendicular

\[
\mathbf{k}_R \cdot \mathbf{k}_I = 0
\]

\[
(\text{17})
\]

Therefore for real \( k \) we have in general

\[
\mathbf{k} = \mathbf{k}_R + i\mathbf{k}_I = k \hat{\mathbf{k}} = k(\mathbf{k}_R/k + i\mathbf{k}_I/k)
\]

\[
(\text{18})
\]

with a complex directional unit vector \( \hat{\mathbf{k}} = (\mathbf{k}_R/k + i\mathbf{k}_I/k) \).

Inasmuch as (15) is a solution of the Maxwell system (1), \( \mathbf{E} \) and \( \mathbf{H} \) satisfy the corresponding
Helmholtz equations (12), e.g.,

\[
(\partial_x^2 + k^2)E(r) = \mathbf{e}_0 \sum_{n=0}^{\infty} (ik)^n (\hat{\mathbf{k}} \cdot \hat{\mathbf{r}})^n / n! = 0
\]

\[
(\partial_x^2 + k^2)E(x) = \mathbf{e}_0 \sum_{n=0}^{\infty} (ik)^n (d_x^2 + k^2)x^n / n! = 0, \quad k = k\hat{x}
\]

\[
(\text{19})
\]

In order that (19) be satisfied as an identity, we have to equate powers of the variable \( x \) (not \( ik \))
and re-index terms accordingly. Thus

\[
\sum_{n=0}^{\infty} (ik)^n d_x^n x^n / n! = \sum_{n=0}^{\infty} (ik)^n n(n-1)x^{n-2} / n!
\]

\[
(\text{20})
\]

where it is noted that in (20) \( n(n-1) \) vanishes for \( n = 0, n = 1 \), allowing the second sum-
mation to start at \( n = 0 \). This treatment also agrees with the prototype scheme\(^1\) mentioned
above.

In a trivial manner we rewrite (19) and (20), and obtain a recurrence relation

\[
E(x) = \mathbf{e}_0 \sum_{n=0}^{\infty} (ik)^n \mathbf{E}_n(x) / n!, \quad \mathbf{E}_n(x) = \hat{\mathbf{e}} x^n
\]

\[
d_x^2 \mathbf{E}_n(x) = n(n-1)\mathbf{E}_{n-2}(x)
\]

\[
E(r) = \mathbf{e}_0 \sum_{n=0}^{\infty} (ik)^n \mathbf{E}_n(r) / n!, \quad \mathbf{E}_n(r) = \hat{\mathbf{e}} (\mathbf{k} \cdot \hat{\mathbf{r}})^n
\]

\[
\partial_r^2 \mathbf{E}_n(r) = n(n-1)\mathbf{E}_{n-2}(r)
\]

\[
(\text{21})
\]

with partial waves in (21) denoted by \( \mathbf{E}_n \).

This elementary example provides the key for generalizing to arbitrary wave fields. The
elementary plane wave can be used as a basis for defining arbitrary wave fields in terms of
Sommerfeld-type integrals.\(^7\)–\(^{12}\) Whether the integral denotes outgoing, incoming, or standing
waves, depends on the specific integration contour \( C \). In general this contour is complex,
prescribing complex directions of propagation for inhomogeneous waves, as explained in (16)-(18).

Accordingly, an arbitrary wave-function is represented as a plane-wave integral of the form

$$E(r) = e_0 \int_C e^{ik \cdot r} g(\hat{k}) d\Omega_{\hat{k}}$$

with $d\Omega_{\hat{k}}$ in (22) indicating the integration over all directions prescribed by the contour $C$, and $g(\hat{k})$ being the amplitude associated with the wave propagating in direction $\hat{k}$. For example, for the two- and three-dimensional cases we have

$$\int_C d\Omega_{\hat{k}} = \frac{1}{\pi} \int_{\beta=-\pi/2 + i\infty}^{\beta=\pi/2 - i\infty} d\beta$$

$$\int_C d\Omega_{\hat{k}} = \frac{1}{2\pi} \int_{\beta=-\pi}^{\beta=\pi} \int_{\alpha=\pi/2 - i\infty}^{\alpha=\pi/2 + i\infty} \sin \alpha d\alpha$$

respectively, with $\beta$ and $\alpha$ denoting complex azimuthal and polar angles respectively.\textsuperscript{9-12}

In view of the Taylor expansion (15), which can also be continued into the complex domain, (23) can be recast in a series of partial waves

$$E(r) = e_0 \sum_{n=0}^{\infty} (ik)^n E_n(r)/n!$$

where in (24) the series representation, derived from a superposition of plane waves, satisfies the condition of analyticity discussed after (15), i.e., that $E(r)$, considered as a complex function of $k$, is regular in the vicinity of $k = 0$.

Applying the same argument that led from (19) to (21), we arrive at the identical recurrence relation

$$\partial_r^2 E_n(r) = n(n - 1) E_{n-2}(r)$$

4. LOW FREQUENCY EXPANSIONS FOR CONSISTENT MAXWELL SYSTEMS

Considering (24) and the first consistent Maxwell system (13) gives

$$\partial_r \cdot E(r) = e_0 \sum_{n=0}^{\infty} (ik)^n \partial_r \cdot E_n(r)/n! = 0$$

On account of our maxim that (26) is a constant-coefficient series in $ik$, we cannot claim that a necessary condition for (26) to be valid is that each term in the series individually vanishes. However, since $\partial_r \cdot E(r) = 0$ vanishes for arbitrary $r$, a sufficient condition is that the series vanishes term by term. Thus (26) is implied by

$$\partial_r \cdot E_n(r) = 0$$

The converse argument does not follow: (26) does not imply (27) in general. This key argument, henceforth referred to as the sufficiency condition, will be met in similar circumstances
below. Let us now attempt to define $H(r)$ according to (3) and seek for the relation between the partial fields $E_n$ and $H_n$. Subject to the first consistent Maxwell system (13) we have

$$\begin{align*}
\partial_r \times E &= e_0 \sum_{n=0}^{\infty} (ik)^n \partial_r \times E_n(r)/n! \\
&= i \omega \mu H = i \omega \mu h_0 \sum_{n=0}^{\infty} (ik)^n H_n(r)/n!, \quad e_0/h_0 = Z
\end{align*}$$

implying according to the sufficiency condition that

$$\partial_r \times E_n(r) = ik H_n(r)$$

Summarizing, the partial-wave representation of the first consistent Maxwell system (13) is given by

$$\begin{align*}
\partial_r^2 E_n(r) &= n(n-1) E_{n-2}(r), \quad \partial_r \times E_n(r) = ik H_n(r) \\
\partial_r \cdot E_n(r) &= 0, \quad \partial_r \cdot H_n(r) = 0
\end{align*}$$

Using the same line of arguments, the second consistent Maxwell system (14) results in the partial-wave representation

$$\begin{align*}
\partial_r^2 H_n(r) &= n(n-1) H_{n-2}(r), \quad \partial_r \times H_n(r) = -ik E_n(r) \\
\partial_r \cdot E_n(r) &= 0, \quad \partial_r \cdot H_n(r) = 0
\end{align*}$$

5. ELECTROMAGNETIC SCATTERING FROM A CYLINDER

This simple scattering problem provided for us a prime motivator for scrutinizing the low-frequency theory. As will be shown below, the fields prescribed by (4) for $n = 0$, namely

$$\begin{align*}
\partial_r \times E_0(r) &= 0, \quad \partial_r \times H_0(r) = 0, \quad \partial_r \cdot E_0(r) = \partial_r \cdot H_0(r) = 0
\end{align*}$$

cannot simultaneously exist as solutions of one and the same scattering problem.

Consider a perfectly conducting infinitely long circular-cylinder of radius $r = a$, with the cylindrical axis $r = 0$ being along the $z$-coordinate. The incident wave is given by (15) and (21), with $\hat{e} = \hat{z}$, $\hat{h} = -\hat{y}$, $\hat{k} = \hat{x}$. This means that we consider here the TM (transverse magnetic) scalar problem. Accordingly we choose (2) and the first consistent Maxwell system (30) to describe the scattered field. Thus the incident and scattered fields are denoted by

$$\begin{align*}
E_i(r) &= \tilde{z} e_0 e^{ikr} = \tilde{z} e_0 e^{ikr \cos \psi} = \tilde{z} e_0 \sum_{n=0}^{\infty} (ik)^n (r \cos \psi)^n/n! \\
E_s(r) &= \tilde{z} e_0 \sum_{n=0}^{\infty} (ik)^n E_{s,n}(r)/n!, \quad \cos \psi = \hat{x} \cdot \hat{r} = \hat{k} \cdot \hat{r}
\end{align*}$$

respectively, with $\psi$ in (33) denoting the azimuthal angle, and $\hat{r}$ pointing away from the cylindrical axis.

The boundary-condition prescribes the vanishing of the tangential electric field at the boundary. For the present case, with fields polarized in the $\hat{z}$ direction

$$\hat{r} \times (E_i(r) + E_s(r)) = 0|_{r=a}$$

According to the sufficiency condition, invoked for arbitrary $\psi$, the series satisfy (34) term by term, or in other words, (34) must be satisfied for any $\psi$. Hence powers of $\cos \psi$ provide the orthogonal basis for the series. In addition to these constraints, $E_s(r)$, in conjunction with the harmonic time variation factor $e^{-i\omega t}$, must describe outgoing waves, i.e., the fields
must satisfy a radiation condition. This will be discussed in the context of the exact solution presented below.

According to (30), for \( n = 0 \) (33) prescribes \( \mathbf{E}_{s,0}(\mathbf{r}) \) as a solution of the Laplace equation, independent of \( \psi \). The electrostatic potential of a uniformly charged infinite line suggests a monopole logarithmic solution, which is the only solution satisfying the above provisos

\[
\mathbf{E}_{s,0}(\mathbf{r}) = \hat{z}A_0 \ln \kappa r, \quad \partial_r \cdot \mathbf{E}_{s,0} = 0, \quad \kappa = 1.
\]  

(35)

where in (35) \( A_0 \) is a constant coefficient and \( \kappa \) is a unit inverse-length constant carrying the physical units in order for the argument of the logarithm function to be dimensionless.

According to (34)

\[
\mathbf{E}_{i,0} + \mathbf{E}_{s,0} = 0 \mid_{r=a}, \quad A_0 = -1/\ln \kappa a, \quad \mathbf{E}_{s,0}(\mathbf{r}) = -\hat{z} \ln(\kappa r)/\ln(\kappa a) \]  

(36)

In accordance with (30), the field \( \mathbf{H}_{s,0} \) associated with (36) is given by

\[
\mathbf{H}_{s,0}(\mathbf{r}) = \partial_r \times \mathbf{E}_{s,0}(\mathbf{r}) = \hat{z}[ikr \ln(\kappa a)] \]  

\[ \partial_r \cdot \mathbf{H}_{s,0} = 0, \quad \partial_r \times \mathbf{H}_{s,0} = 0 \]  

(37)

It is obvious from (37) that \( \partial_r \times \mathbf{H}_{s,0} = 0 \) but \( \partial_r \times \mathbf{E}_{s,0}(\mathbf{r}) \neq 0 \), otherwise we cannot have associated electric and magnetic fields satisfying (30), see also (32) and following remarks.

Comparing (36) and (37), we see that for increasing \( r \) the ratio of field amplitudes behaves like

\[
\frac{\mid \mathbf{E}_{s,0} \mid}{\mid \mathbf{H}_{s,0} \mid} \propto \frac{\mid \ln(\kappa r) \mid}{r} \sim r
\]  

(38)

i.e., the electric field becomes dominant over the magnetic field, as expected.

For the next term \( n = 1 \) , (33) prescribes a factor \( \hat{k} \cdot \hat{r} = \cos \psi \). Hence in order to satisfy the boundary-conditions on the cylinder at \( r = a \) for all \( \psi \), the factor \( \cos \psi \) must feature in the solution. We therefore choose from the electrostatic potential theory repertoire a dipole term

\[
E_{s,1}(\mathbf{r}) = \hat{z}ikA_1 \hat{k} \cdot \hat{r}/r = \hat{z}ikA_1 \cos \psi/r, \quad \partial_r^2 E_{s,1} = 0, \quad \partial_r \cdot E_{s,1} = 0
\]  

(39)

The constant \( A_1 \) is determined from the boundary-condition, hence we have

\[
E_{i,1} + E_{s,1} = 0 \mid_{r=a}, \quad A_1 = -a^2, \quad E_{s,1}(\mathbf{r}) = -\hat{z}ika^2 \cos \psi/r
\]  

(40)

According to (30), the corresponding magnetic field is given by

\[
\mathbf{H}_{s,1}(\mathbf{r}) = -\partial_r \times \hat{z}ika^2 \cos \psi/ikr
\]  

\[ = a^2 \hat{z} \times \partial_r(\cos \psi/r) = (a^2/r^2)(\hat{r} \sin \psi - \hat{\psi} \cos \psi) \]  

(41)

Similarly to (38) we have now

\[
\frac{\mid E_{s,1} \mid}{\mid H_{s,1} \mid} \sim r
\]  

(42)

So once again the electric field becomes dominant over the magnetic for increasing \( r \).
The exact solution for scattering from a perfectly conducting circular cylinder is a well known problem, e.g., see.\textsuperscript{13} The incident wave (33) and scattered wave solution of (12) are stated in terms of cylindrical wave functions
\[
\begin{align*}
E_0(r) &= \hat{z}\varepsilon_0 \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\psi} \\
E_\theta(r) &= \hat{z}\varepsilon_0 \sum_{m=-\infty}^{\infty} i^m a_m H_m(kr) e^{im\psi} \\
H_m &= H_m^{(1)} = J_m + iN_m
\end{align*}
\] (43)

The Hankel functions of the first kind \( H_m = H_m^{(1)} \) are a combination of the nonsingular Bessel functions \( J_m \), and the Neumann functions \( N_m \) which are singular at \( kr = 0 \) (e.g., see\textsuperscript{7}). Combined with the time factor \( e^{-i\omega t} \), outgoing scattered waves are stated in (43).

For cylindrical functions of real integer order we have\textsuperscript{17,18}
\[
Z_m(kr) = (-1)^m Z_m(kr), \quad Z = J, \quad N, \quad H^{(1)}, \quad H^{(2)}
\] (44)
for Bessel, Neumann, and the first and second kind Hankel functions, respectively. The coefficients \( a_m \) in (43) are computed by solving the pertinent boundary value problem prescribed by (34) (e.g., see\textsuperscript{13}). Together with (44) we thus obtain
\[
a_m = a_{-m} = -J_m(ka)/H_m(ka)
\] (45)
For small arguments, formulas for the cylindrical functions near the origin\textsuperscript{7} are exploited, e.g.,\textsuperscript{9} yielding for \( a_0(ka), \quad H_0(kr) \) in (43) for small \( a \) and \( r \)
\[
\begin{align*}
a_0(ka) &\approx -i\pi/[2 \ln(2/(\delta kr))] \approx i\pi/[2 \ln a] \\
H_0(kr) &= J_0(kr) + iN_0(kr) \approx iN_0(kr) \approx -i(2/\pi) \ln(2/(\delta kr)) \\
&\approx i(2/\pi) \ln r \\
a_0(ka)H_0(kr) &\approx -\ln r / \ln a
\end{align*}
\] (46)
where \( \delta \) denotes the Euler constant.\textsuperscript{7} This is exactly the expression given in (36). Summing the terms for \( m = \pm 1 \) in (43), (45), we obtain
\[
\begin{align*}
a_1 = a_{-1} = -J_1(ka)/H_1(ka) &\approx -i\pi(ka)^2/4 \\
H_1(kr) &\approx iN_1(ka) \approx -i(2/\pi kr) \\
i2a_1(ka)H_1(kr) \cos \psi &\approx -ika^2 \cos \psi / r
\end{align*}
\] (47)
in full agreement with (40).

For TE polarization, i.e., having the \( \mathbf{H} \) field polarized along the \( z \)-coordinate, we follow a similar procedure. We use the second consistent Maxwell system (14). The analog of (43) is given by
\[
\begin{align*}
\mathbf{H}_i(r) &= \hat{\mathbf{z}}\varepsilon_0 \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\psi} \\
\mathbf{E}_i(r) &= -\varepsilon_0 \sum_{m=-\infty}^{\infty} i^m e^{im\psi} (\hat{\mathbf{r}} m/r + i\hat{\mathbf{z}} \partial_k r) J_m(kr) \\
\mathbf{H}_s(r) &= \hat{\mathbf{z}}\varepsilon_0 \sum_{m=-\infty}^{\infty} i^m b_m H_m(kr) e^{im\psi} \\
\mathbf{E}_s(r) &= -\varepsilon_0 \sum_{m=-\infty}^{\infty} i^m b_m e^{im\psi} (\hat{\mathbf{r}} m/r + i\hat{\mathbf{z}} \partial_k r) H_m(kr)
\end{align*}
\] (48)
with the same boundary-conditions (34). The analog of (45) is now
\[
b_m = b_{-m} = -\partial_{ka} J_m(ka)/\partial_{ka} H_m(ka)
\] (49)
The details of the expansion of the leading terms of (48), (49), are somewhat more complicated and are omitted.
6. CONCLUDING REMARKS

Low-frequency theory is important for various branches of wave physics. To that end Morse and Feshbach,\(^1\) represent the solution of the Helmholtz equation as series in ascending powers of \(k\) (see p. 1085, equation (9.3.56)). Recurrence relations are then found (equation 9.3.57)), essentially identical to (30) and (31). Other researchers, notably\(^2\)–\(^6\) extended the validity of the series on the assumption that fields can be expanded in \textit{variable-coefficient series} in terms of the inverse wavelength. The wavenumber, \(k\), was treated as a variable. However, this assumption is inconsistent with the frequency \(\omega\), hence also \(k\), being constants characterizing the harmonic time variation and the associated Helmholtz wave equations.

The present study does not invoke variable \(k\) and variable-coefficient series properties. In fact, it has been argued and verified by a simple example of scattering by a perfectly-conducting circular cylinder that the old theory does not lead to correct results.

Instead, our study is based on the concept of consistent systems, which are equivalent to the basic physical models in electromagnetics, acoustics, and elasticity. The present consistent systems contain Helmholtz equations for which series manipulation and re-labeling of indices are shown to hold. In expressions relating more than one field, invoking the idea of "equating powers of \(ik\)" implies variable \(ik\), therefore variable-coefficient series, as done previously.\(^2\)–\(^5\) The present arguments rely on Sommerfeld-type plane-wave integrals, which allow for a consistent derivation of the pertinent series. In general the contour of integration is complex, involving inhomogeneous plane-waves possessing complex propagation vectors \(k\) but real values of \(k^2 = k \cdot k\), a situation that can be achieved even in non-absorbing media, such as free-space (vacuum), when the real and imaginary components are perpendicular, i.e., \(k_R \cdot k_I = 0\).

Further work is planned in comparing the low-frequency results with exact solutions, e.g., the Mie solution (e.g., see\(^7\)) for scattering by a sphere. This will involve more complicated special functions associated with vector spherical-waves.

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