SCATTERING BY TIME VARYING OBSTACLES

D. CENSOR

Department of Environmental Sciences,
Tel-Aviv University, Ramat-Aviv, Israel

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The present paper considers scattering problems involving impenetrable obstacles which deform as a function of time. The method can be applied to practical vibration problems involving machinery with sliding parts or with flexible components changing their form as a function of time.

In particular, the theory is applied to periodically varying shapes which can be conveniently related to circular cylinders and spheres. Special cases of interest are discussed: e.g., radially and lineally vibrating objects, and rotating ellipses of small eccentricity. Explicit results are given for small deformations and the derivation of higher order effects is delineated.

Although the incident wave is monochromatic, the scattered wave has a spectrum of frequencies. The main effect is the production of two sidebands whose spectral and spatial structure might provide a signature for the special mode of motion at hand.

1. INTRODUCTION

The class of problem discussed here involves scattering by objects presenting time-dependent boundary conditions. Instead of considering the differential equations, as done by Lam [1] for the expanding electromagnetic sphere, the present method starts with the solution of the elliptic (Helmholtz) wave equation. A general spectral representation is constructed, and the application of the time-dependent boundary conditions determines the spectrum for the problem at hand. This has been done by Censor [2] for the one-dimensional case. For two and three dimensions solutions have been derived previously [3, 4] for expanding objects, in the vicinity of an arbitrarily chosen time, \( t = 0 \).

Here we consider circular cylinders and spheres deforming as a function of time. Application of the boundary conditions leads to an integral equation which for the cases considered here can be solved by a simple perturbation scheme. It is shown that the motion of the scatterer modulates the wave field, producing new frequencies.

For simplicity harmonically vibrating obstacles and a few special modes of motion have been chosen. However, the method is not restricted to these cases only, and is easily adaptable to other configurations. Therefore it could be applied to more practical vibration problems where moving or time varying bodies are present.

2. STATEMENT OF THE PROBLEM

Scattering of waves by impenetrable time varying cylinders and spheres is considered. An arbitrary incident wave can be decomposed into a superposition (sum or integral) of waves harmonic in time (by using Fourier analysis) and space (by using a plane wave representation); hence we focus our attention on incident waves of the form

\[ U_i = A \exp \left( i k \cdot R - i \omega t \right), \]

where \( A \) is the amplitude, \( R \) is the radius vector, \( k \) is the propagation vector, \( k = \omega/c \), where \( \omega \) is the (angular) frequency and \( c \) is the wave speed; \( t \) is the time.
In general the scattered field $U_s(R, t)$ is non-monochromatic, hence it is represented as a superposition of solutions of the Helmholtz wave equation

$$U_s(R, t) = \int_{-\infty}^{\infty} U_\nu(R) \exp (-ivt) \, dv,$$  \hspace{1cm} (2)

such that the integrand is the solution for frequency $\nu$ and the integration is over all possible frequencies. The partial waves in the integrand (2) are chosen such that the pertinent two- or three-dimensional radiation condition is satisfied.

For the subsequent cases it suffices to consider scatterers whose surfaces are described analytically by smooth single valued functions of the form

$$r = r(\phi, z, t),$$
$$R = R(\theta, \phi, t),$$  \hspace{1cm} (3)

for the two- and three-dimensional cases, respectively. Here $r$ is the distance from the cylindrical axis $z$, and $\phi$ is the angle measured in a plane perpendicular to $z$. For the three-dimensional case $\theta$ is the polar angle, measured off the polar axis $z$, and $\phi$ is the same as before.

At the surface of the scatterer the combined wave $U_i + U_s$ satisfies prescribed boundary conditions. This is symbolically denoted by an operator $O$, which may involve differentiation. Thus we have

$$O(U_i + U_s) = 0 \big|_{\text{at the surface}}$$  \hspace{1cm} (4)

which turns out to be an integral equation, involving the time as a parameter. The solution of equation (4) determines $U_s$. The method of solving equation (4) will be best understood from the membrane problem, given below.

3. PROBLEM OF THE MEMBRANE

As an example of a simple two-dimensional problem, consider the case of transverse waves, propagating on an ideal membrane and scattered by a time varying impenetrable deformed circle.

The general solution of the Helmholtz wave equation is represented as

$$\sum_{m = -\infty}^{\infty} i^m a_m(\omega) Z_m(kr) \exp(i m \phi - i\omega t),$$  \hspace{1cm} (5)

where $Z_m$ is a solution of the Bessel differential equation. The scattering coefficients $a_m(\omega)$ are to be determined by the condition that the displacement vanishes at the boundary. While the coefficients $a_m(\omega)$ depend on the parameters of the problem, they must be independent of $r, \phi, t$; otherwise, in general, equation (5) is not a solution of the wave equation.

The incident wave is chosen as in equation (1) with $A = 1$ and describes the transverse displacement. Recasting equation (1) in circular cylindrical wave functions yields equation (5) with $a_m Z_m$ replaced by $J_m$, the non-singular Bessel functions. For an arbitrary frequency, $\nu$, in equation (5) $\omega, k$ are replaced by $\nu, \nu/c$, respectively. For the scattered field, $Z_m$ in equation (5) is replaced by $H_m = H_m^{(1)}$, the Hankel function of the first kind, which in conjunction with $\exp(-i\nu t)$ satisfies the radiation condition (e.g., Sommerfeld’s
radiation condition), such that the scattered field behaves as an outgoing wave of properly diminishing amplitude. For arbitrary time variation the analog of equation (2) is

\[ U_s = \int_{-\infty}^{\infty} dv \sum_m i^m a_m(v) H_m \left( \frac{vr}{c} \right) \exp(imm - ivt), \]  

where \( a_m(v) \) will be determined by the relevant geometry.

The scatterer is a deformed circle such that

\[ r = a[1 + f(\phi, t)], \]  

where \( a \) is the unperturbed radius. For \( r \) to be always non-negative \( |f| \leq 1 \) must be assumed.

At the boundary (7) the displacement vanishes; hence equation (4) becomes

\[ \sum_m i^m \exp(im\phi) \left\{ J_m \left( \frac{\omega a[1 + f(\phi, t)]}{c} \right) \exp(-i\omega t) \right\} + \int_{-\infty}^{\infty} dv a_m(v) H_m \left( \frac{va[1 + f(\phi, t)]}{c} \right) \exp(-ivt) = 0. \]  

As a first example, let the circle move in radial harmonic motion according to

\[ f(t) = \varepsilon \sin \Omega t, \]  

where \( \varepsilon \) is the perturbation parameter (see Figure 1). Since equation (9) is independent of \( \phi \), equation (8) can be solved for each mode separately. Expand the cylindrical functions in a Taylor series, and let

\[ a_m(v) = A_m(v) + \varepsilon B_m(v) + \ldots, \]  

and collect terms of the same power in \( \varepsilon \). Then, by inspection of equation (8),

\[ A_m(v) = \frac{-\delta(v - \omega) J_m(va/c)}{H_m(va/c)}. \]
Substitution of equation (11) in equation (6) yields the well-known result for the unper-
turbed rigid circle. For terms of first order in $\varepsilon$ we get

$$
g(t) = \int_{-\infty}^{\infty} dv \, G(v) \exp (-ivt),$$

$$
g(t) = g_1(t) g_2(\rho),$$

$$
g_1(t) = \sin \Omega t \exp (-i \omega t),$$

$$
g_2(\rho) = \rho \left[ \frac{J_m(\rho) H_m'(\rho)}{H_m(\rho)} - J_m'(\rho) \right],$$

$$
\rho = \omega a/c,$$

$$
G(v) = B_m(v) H_m \left( \frac{va}{c} \right), \quad (12)$$

where the prime denotes differentiation with respect to the argument. When the
Wronskian relations for cylindrical functions are used \cite{5}, $g_2(\rho)$ simplifies to

$$
g_2(\rho) = \frac{2i}{\pi H_m(\rho)}. \quad (13)$$

Since equation (12) has the structure of a Fourier transform, by inversion we obtain

$$
G(v) = \frac{g_2(\rho)}{2\pi} \int_{-\infty}^{\infty} g_1(t) \exp (ivt) \, dt. \quad (14)$$

This yields

$$
B_m(v) = \left[ \pi H_m(\rho) H_m \left( \frac{va}{c} \right) \right]^{-1} [\delta(v - \omega + \Omega) - \delta(v - \omega - \Omega)]. \quad (15)$$

By substituting $\varepsilon B_m(v)$ into equation (6), the first-order correction due to the motion is
found. Thus two sidebands at frequencies $\omega_{\pm} = \omega \pm \Omega$ are produced. If instead of a single
frequency $\Omega$ we have a spectrum, this will appear symmetrically with respect to $\omega$ in the
spectrum of the scattered wave. Higher order powers of $\varepsilon$ produce additional sidebands.
It is easy to show that these cases too lead to Fourier transforms and their inverses.
Therefore, although the expressions become cumbersome and are not given here, the
procedure for solving the problem within a desired accuracy in terms of powers of $\varepsilon$ is
straightforward. In general higher order approximations produce frequencies farther
away from $\omega$, but also contribute to spectral components nearer to $\omega$; therefore it might
be more complicated to sound the motion by analyzing the spectrum of the scattered
wave. For simple harmonic vibration and frequencies $\omega > \Omega$ such that $H_m(\omega \pm a/c) \approx H_m(\rho),$
the two main sidebands produce a beat note at frequency $\omega$, varying in amplitude according
to the modulating frequency $\Omega$.

Another interesting case is provided by

$$
r = a[1 + \varepsilon f(t) \cos (\phi - \alpha)] + O(\varepsilon^2), \quad (16)$$

amounting to a deformation which leaves the surface a perfect circle, moving lineally along
the line $\phi = \alpha$, as shown in Figure 2. It must be stressed that this interpretation is valid
only for impenetrable surfaces, where the tangential motion of a point on the surface has
no effect.
The modes of motion leaving the scatterer as a perfect circle can be generalized. In fact, by combining \( \sin \phi \) and \( \cos \phi \) with \( \sin n\Omega t \) and \( \cos n\Omega t \), the surface can be made to move as an undistorted circle whose center traces various Lissajous figures (as one sees on an oscilloscope when feeding time harmonics to the \( X \) and \( Y \) deflecting plates): for example, a circle, a figure-eight, etc.

By solving the simple problem of lineal motion, with \( \alpha = 0 \), once again \( A_m \) is obtained as in equation (11). The presence of \( \cos \phi \) as a factor in some of the terms of order \( \varepsilon \) requires that the series be reshuffled before the orthogonality of \( \exp (i m \phi) \) can be exploited. Working along the same lines as before, we get

\[
B_m(v) = -i \frac{\delta(v - \omega + \Omega) - \delta(v - \omega - \Omega)}{2\pi H_m(va/c)} \left[ \frac{1}{H_{m-1}(\rho)} - \frac{1}{H_{m+1}(\rho)} \right].
\]

(17)

Once again it is seen that the first-order effect is the production of two sidebands. Because of the dependence of \( \phi \) in equation (16) we have now mode coupling, in the sense that \( B_m \) depends on \( m + 1, m - 1 \). All modes of motion except the radial mode show this behaviour.

Another interesting mode of motion is exemplified by

\[
r = a(1 + \varepsilon \sin \Omega t \cos 2\phi),
\]

(18)

describing, as shown in Figure 3, an ellipse with oscillating minor and major axes, such that one increases as the other decreases, and \textit{vice versa}. The generalization of this kind of motion would lead to an arbitrary standing wave pattern on the circumference of the circle. The presence of \( \cos 2\phi \) produces coupling between modes \( m \) and \( m \pm 2 \).
For this case $A_n(v)$ is the same as before and $B_n(v)$ is given by equation (17) with the expression in brackets replaced by

$$-i\left(\frac{1}{H_{m+2}(\rho)} + \frac{1}{H_{m-2}(\rho)}\right).$$

(19)

![Figure 4. Geometry for a rotating ellipse.](image)

The last example to be discussed is the case of a rotating ellipse, described by

$$r = a[1 + \varepsilon \cos(\Omega t - 2\phi)].$$

(20)

The generalization of this mode is the case of a rotating corrugated circle. For the present case $A_n(v)$ is the same as before, and it is found that

$$B_n(v) = -\frac{i}{\pi H_n(va/c)} \left[\frac{\delta(v - \omega + \Omega)}{H_{m+2}(\rho)} + \frac{\delta(v - \omega - \Omega)}{H_{m-2}(\rho)}\right].$$

(21)

(See Figure 4.)

4. PROBLEM OF THE CYLINDER

In this section the problem of scattering by a time-varying cylinder is considered. In the case of the membrane, the question of interaction of the scatterer with the medium supporting the waves did not arise. In contradistinction, here the medium surrounding the cylinder is set into motion. Therefore, in addition to the scattered wave we encounter the radiation field produced by the moving body. Henceforth it will be assumed that the requirements of linear acoustics are not violated, which implies that the two phenomena are superposed. Especially for $\Omega$ small with respect to $\omega$, it is reasonable to assume that the energy carried by the scattered field is dominant; hence only this phenomenon will be investigated.

For acoustical waves let equations (1) and (2) describe the pressure. The simplest case is provided by normal incidence on a free cylindrical surface. Since the pressure vanishes at the surface, all the results of the previous section are applicable. The other extreme case is a cylinder of a medium much harder than the surrounding one, which we shall call rigid, although we consider its time-dependent deformations. For a rigid cylinder at rest the boundary condition is that the normal acoustical velocity or displacement vanishes at the surface. For moving boundaries these two conditions are not equivalent. However, the consensus seems to be that the more basic condition is the vanishing of the normal displacement. For this case the operator $\partial$ in equation (4) is

$$v^{-2} \mathbf{n} \cdot \nabla,$$

(22)

where $v^{-2}$ results from a double time integration, $v$ being the frequency of the spectral component in question, for the incident wave $v = \omega$; $\mathbf{n}$ is a unit vector perpendicular to the perturbed surface.
For all modes of motion, for the unperturbed cylinder the well-known result is obtained:

\[ A_m(v) = -\frac{\delta(v - \omega) J_m'(\rho) H_m''(\rho)}{H_m'(\rho)}. \]  \hfill (23)

The case of radial vibrations (see equations (7) and (9)) is simple because \( \hat{n} = \hat{r} \), yielding

\[ B_m(v) = \left[ 2i H_m' \left( \frac{\nu a}{c} \right) \right]^{-1} \nu g_m(\rho) \left[ \delta(v - \omega + \Omega) - \delta(v - \omega - \Omega) \right], \]

\[ g_m(\rho) = \left[ \frac{\rho}{H_m''(\rho)} \right] \left[ H_m''(\rho) J_m'(\rho) - J_m''(\rho) H_m'(\rho) \right]. \]  \hfill (24)

This can be further compacted by using the Basset relation [6]:

\[ J_m'(\rho) H_m''(\rho) - H_m'(\rho) J_m''(\rho) = \left( \frac{2}{i\nu\rho} \right) \left( 1 - \frac{m^2}{\rho^2} \right). \]  \hfill (25)

We now turn our attention to the case of a (acoustically) rigid cylinder, normal incidence and lineal motion as described by Figure 2. Here we have

\[ -\nu^{-2} \hat{n} \cdot \nabla = -\nu^2 \left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial \phi} \left( c/\rho \sin \Omega t \right) \right], \]  \hfill (26)

which leads to

\[ B_m(v) = \left[ 4\nu H_m' \left( \frac{\nu a}{c} \right) \right]^{-1} \left( \alpha_{m-1} - \alpha_{m+1} + \beta_{m-1} + \beta_{m+1} \right) \left[ \delta(v - \omega + \Omega) - \delta(v - \omega - \Omega) \right], \]

\[ \alpha_m = \frac{\rho}{\nu} \left[ J_m''(\rho) - \frac{H_m''(\rho) J_m'(\rho)}{H_m'(\rho)} \right], \]

\[ \beta_m = \frac{m}{\rho \nu} \left[ J_m(\rho) - \frac{H_m(\rho) J_m'(\rho)}{H_m'(\rho)} \right], \]  \hfill (27)

which again can be compacted by using Basset relations.
Without going into the details for other modes of motion similar to the membrane cases, it is concluded that again the production of two sidebands at \( \omega_{\pm} = \omega \pm \Omega \) is the main characteristic of this class of scattering problems.

For the cylinder problem we have another class of modes of motion: namely, when the motion is a function of \( z \), modes along the axis of the cylinder.

Let us choose, as an example, the case of a standing wave pattern given by

\[
\dd r = a(1 + \varepsilon \sin \Omega t \cos k \zeta).
\]  

(28)

(See Figure 5.) Even though the incident wave is taken as propagating normally with respect to the \( z \)-axis, in the scattered wave we must allow for waves in other directions. Hence we construct

\[
U_s = \int_0^a d\zeta \int_{-\infty}^{\infty} d\nu \sum_m i^m \alpha_m(\nu, \alpha) H_m(\lambda r) \exp [im\phi + iA(\nu, \alpha) z - ivt],
\]

\[
\lambda = \frac{v}{c} \sin \alpha;
\]

\[
A = \frac{v}{c} \cos \alpha,
\]

(29)

where \( \alpha \) is measured off the \( z \)-axis.

The unit vector normal with respect to the perturbed surface described by equation (28) is

\[
\hat{n} = \hat{r} + 2 \frac{a}{c} \varepsilon \sin \Omega t \sin k \zeta;
\]

(30)

hence

\[
-\frac{v^2}{c} \hat{n} \cdot \nabla = -\frac{v^2}{c} \left[ \frac{\partial}{\partial r} + \frac{\partial}{\partial \zeta} \frac{a}{c} \varepsilon \sin \Omega t \sin k \zeta \right].
\]

(31)

Application of the boundary condition to \( U_t + U_s \) yields for \( A_m \) the result given by equation (23), multiplied by \( \delta(\alpha - \pi/2) \). Upon substitution in equation (29) this selects the direction normal to the axis for the scattered wave of the zeroth order in \( \varepsilon \). The terms of order \( \varepsilon \) prescribe two values of \( \alpha \) such that

\[
B_m(\nu, \alpha) = B_m^+(\nu) \delta[\alpha - \arccos \kappa c/\nu] + B_m^-(\nu) \delta[\alpha + \arccos (\kappa c/\nu)],
\]

(32)

and correspondingly we have

\[
\lambda^{\pm} \equiv \lambda_1 = \frac{1 - \cos^2(\kappa c/\nu)}{1 + \cos^2(\kappa c/\nu)} \right)^{1/2}.
\]

(33)

Finally

\[
B_m^{\pm}(\nu) = \frac{\nu^2 \left[ J_m'(\rho) H_m''(\rho)/H_m'(\rho) - J_m''(\rho) \right] \left[ \delta(\nu - \omega + \Omega) - \delta(\nu - \omega - \Omega) \right]}{\cos \lambda_1 H_m(\lambda_1 a)},
\]

(34)

which again can be compacted by using Basset relations. Consequently it is seen that we again have two sidebands at \( \omega_{\pm} \). In addition, there are two obliquely scattered waves, that have not been present in the exciting wave.
5. PROBLEM OF THE SPHERE

In spherical coordinates the general solution of the Helmholtz equation is

\[ Y_{nm}(\theta, \phi) = P_{nm}^m(\cos \theta) \exp(-im\phi), \]

where the \( z_n \) are spherical Bessel or Hankel functions, the \( Y_{nm}^m \) are spherical harmonics involving \( P_{nm}^m(\cos \theta) \), the associated Legendre functions, and the \( a_{nm} \) are coefficients, to be determined by boundary conditions. For the incident wave, propagating in the direction \( \theta = 0 \) we take equation (35) with \( m = 0 \) and \( a_{n0} z_n = j_n \), where \( j_n \) are the non-singular spherical Bessel functions. For the scattered wave a spectrum is considered, as before, and in equation (35) \( z_n \) is replaced by \( h_n \), the spherical Hankel function of the first kind.

For radial isotropic motion as described by equation (7) with \( R \) replacing \( r \), and equation (9), we get for \( A_n \) the structure of equation (11) with \( m \) replaced by \( n \) and \( J, H \) replaced by \( j, h \), respectively.

The same changes made in equation (12) yield \( B_n(v) \), the first-order correction. Similarly to equation (13), we have here

\[ g_2(\rho) = \frac{1}{h_n(\rho)}, \]

and this changes equation (15) correspondingly. Hence essentially the same results are obtained for the soft sphere as for the membrane. For the hard sphere, from equation (23) \( A_n(v) \) is obtained with \( m \) replaced by \( n \), and \( J, H \) replaced by \( j, h \). The analog of equation (24) yields \( B_n(v) \) by performing the same changes.

Modes of motion independent of \( \phi \) will bear similarity to the cylinder modes independent of \( z \). Thus, for lineal harmonic motion of a rigid sphere along \( \theta = 0 \),

\[ \mathbf{n} \cdot \nabla = \frac{\partial}{\partial R} + \frac{\partial}{\partial \theta} \left( \frac{\varepsilon}{R} \right) \sin \Omega t \sin \theta. \]

Now we use the recurrence relations

\[ (2n + 1) \sin \theta \frac{\partial P_n}{\partial \theta} = n(n + 1) (P_{n+1} - P_{n-1}), \]

\[ (2n + 1) \cos \theta P_n = (n + 1) P_{n+1} + nP_{n-1}. \]

This yields for the first-order correction

\[ B_n(v) = \left[ \frac{C_n v}{h_n'(v \alpha/c)} \right] \left[ \delta(v - \omega + \Omega) - \delta(v - \omega - \Omega) \right], \]

\[ C_n = \frac{[n\alpha_{n-1} - (n + 1) \alpha_{n+1} + n(n - 1) \beta_{n-1} + (n + 1)(n + 2) \beta_{n+1}]}{2(2n + 1)}, \]

\[ \alpha_n = \left[ \frac{j_n''(\rho) - h_n''(\rho)j_n'(\rho)}{h_n'(\rho)} \right] \frac{\rho}{\omega}, \]

\[ \beta_n = \left[ \frac{j_n(\rho) - j_n'(\rho)h_n(\rho)}{\rho \omega} \right]. \]
Finally we have to consider arbitrary modes depending also on $\phi$. As a simple case consider the soft sphere and motion described by

$$R = a[1 + \varepsilon \sin \Omega t \cos \phi]. \quad (40)$$

Similarly to the cylinder case with motion depending on the $z$-coordinate, here the full dependence on $\phi$ is retained in the scattered wave. The condition that $U_i + U_s = 0$ at $R$ given by equation (40) yields $A_n$ as before. Then to the first order in $\varepsilon$ we have to solve

$$\sum_{n=0}^{\infty} i^n(2n + 1) \left\{ \exp(-i\omega t) \left[ j'_n(\rho) - \frac{j_n(\rho) h'_n(\rho)}{h_n(\rho)} \right] \rho a \cos \phi P_n(\cos \theta) ight\} + \int_{-\infty}^{\infty} dv \sum_{m=-n}^{n} \frac{(n - m)!}{(n + m)!} B_{nm} h_n(\frac{va}{c}) Y_n^m(\theta, \phi) \exp(-ivt) = 0. \quad (41)$$

Exploiting the fact that an arbitrary function $g(\theta, \phi)$ can be recast as a series in spherical harmonics, we have to find the coefficients $C_{pq}^n$ in the expansion

$$\cos \phi P_n(\cos \theta) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} C_{pq}^n P_p^{q}(\cos \theta) \exp(iq\phi). \quad (42)$$

Evidently only $q = \pm 1$ is admitted, and by exploiting the orthogonality properties of $P_n^m, C_{p,1}^n, C_{p,-1}^n$ are found according to

$$C_{p,\pm 1}^n = \frac{(2p + 1)(p \mp 1)!}{4(p \pm 1)!} \int_{0}^{\pi} P_p^{\pm 1}(\cos \theta) P_n(\cos \theta) \sin \theta \, d\theta. \quad (43)$$

Without going further into this problem it is seen that the reflected wave will have a $\phi$ dependence according to $\exp(\pm i\phi)$, which is a result of the motion. Again the spectrum contains $\omega_\pm$ as before.

REFERENCES