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VELOCITY EFFECTS IN SCATTERING FROM EXPANDING BUBBLES

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The problem of scattering of plane sound waves from radially expanding cylinders and spheres is considered. The problem contains two elements: propagation in radially flowing media in the vicinity of the scatterers, and the effect of the motion of the boundaries on the scattered waves. The problem is solved within the framework of linear acoustics and small velocities of the boundaries. The explicit solutions in terms of the pertinent special functions are obtained by perturbing the time around an arbitrary origin, $t = 0$; therefore the full time history can only be established by piecing together results for various instances, redefining $t = 0$ for each case.

It is hoped that the results might help in probing underwater explosions and the ensuing pulsating gas bubbles.

1. INTRODUCTION

Usually (see for example Urick [1], who also cites many references), underwater explosions are probed by recording the pressure waves produced by the explosion, and the pulsating gas bubble which sometimes exists afterwards. The following analysis is concerned with scattering of sound waves by such configurations. This might be of importance for studying these phenomena: e.g., the size and surface speed of the expanding or contracting bubbles.

Theoretically the problem is of interest for several reasons. Analogous problems for electromagnetic waves have been solved [2], and it is interesting to compare the theoretical model used there with that for the present acoustical case. The present problem involves acoustics in the presence of non-uniformly moving media, which poses many difficulties and is essentially an open problem. For some relevant references see Censor [3], and Censor and Aboudi [4].

In the present analysis it is assumed that the scatterer is defined by means of an expanding or contracting surface. The motion of the surface displaces the surrounding medium radially. Thus one is dealing with propagation of sound waves in radially moving media. The problem is considered as quasi-stationary in the sense that the change in the velocity during the period of observation is negligible. In view of the high contrast between the surrounding fluid and the gas in the interior region, the boundary condition is taken as for a cavity: i.e., the acoustical pressure vanishes at the surface.

The incident wave is chosen as a monochromatic plane sound wave. This is compatible with a localized coherent source, e.g., sonar, at a large distance from the scatterer. The field scattered from a boundary in motion, except for the ideal case of a plane interface, is always time-dependent: i.e., it contains additional frequencies so that it is non-monochromatic. Consequently a concise solution would be possible only in the regime of the hyperbolic wave equation. However, the present analysis, which is correct only to the first order in the velocity, makes it possible to use the solution of the elliptic (Helmholtz) wave equation, by assigning different frequency shifts for various modes.

We start by considering the problem of sound wave propagation in radially moving media. The problem is solved to the first order in the ambient and acoustical velocities. Then the scattering problem is considered, yielding the modal components.

2. PROPAGATION IN RADIALLY FLOWING MEDIA

Within the framework of small amplitude linear acoustics, a first-order formalism is considered for radially moving media.

The equation of motion is

$$\nabla P + \rho(\mathrm{d}/\mathrm{d}t)\mathbf{v} = 0, \tag{1}$$

where P is the total pressure, consisting of the static pressure P_0, and the perturbation p; ρ is the ambient density, and $\mathbf{v}$ is the total velocity, consisting of $\mathbf{v}_0$, the ambient velocity, and the acoustic perturbation $\mathbf{u}$. The perturbation in ρ is neglected, since the fluid's compressibility is considered to be very low. To the first order in $\mathbf{u}$ and p, the time-dependent quantities in equation (1) yield

$$\begin{gathered}\nabla p/\rho + \partial_t \mathbf{u} + (\mathbf{v}_0\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{v}_0 = 0,\\ \partial_t = \partial/\partial t.\end{gathered} \tag{2}$$

For a two-, and three-dimensional radial (almost) incompressible flow we take

$$\begin{gathered}\mathbf{v}_0 = (V/r)\hat{\mathbf{r}},\\ \mathbf{v}_0 = (V/r^2)\hat{\mathbf{r}},\end{gathered} \tag{3}$$

respectively, such that $\nabla\cdot\mathbf{v}_0 = 0$, where V is a proportionality factor, numerically equal to $\mathbf{v}_0$ at $r = 1$ and $\hat{\mathbf{r}}$ in equation (3) is a unit vector in the radial direction, in either a polar, or a spherical coordinate system, respectively.

In a local frame of reference moving with a velocity $\mathbf{v}_0$ (as seen from the laboratory), the ambient velocity vanishes. For the acoustical velocity and pressure the equation of continuity yields (e.g., see Morse and Ingard [5] p. 243)

$$c^{-2}\partial_t p + \rho\nabla\cdot\mathbf{u} = 0, \tag{4}$$

where c is the speed of propagation in the medium at rest. Applying the material derivative, which is tantamount to an instantaneous local Galilean transformation, in the frame of reference of the observer in the laboratory we have

$$c^{-2}(\partial_t + \mathbf{v}_0\cdot\nabla)p + \rho\nabla\cdot\mathbf{u} = 0. \tag{5}$$

Thus, through equations (4) and (5), the parameter c, the speed of propagation in the medium at rest, has been introduced.

Consider first the two-dimensional flow, with sound waves independent of the axial coordinate z. This case, although less realistic, is somewhat simpler, mathematically. Accordingly, equation (2), the first line of equation (3) and equation (5) yield three scalar differential equations. For angular dependence $\mathrm{e}^{im\phi}$ and time dependence $\mathrm{e}^{-i\omega t}$ one obtains

$$\begin{gathered}-i\omega c^{-2}[1 + (iV/\omega r)\,\partial_r]p + \rho(\partial_r + r^{-1})\,u_r + (\rho/r)\,imu_\phi = 0,\\ \partial_r p - i\omega\rho[1 + (iV/\omega r)\,\partial_r - (iV/\omega r^2)]\,u_r = 0,\\ imp/r - i\omega\rho[1 + (iV/\omega r)\,\partial_r + (iV/\omega r^2)]\,u_\phi = 0.\end{gathered} \tag{6}$$

These are three first-order, coupled, linear, ordinary differential equations, and ∂_r can be replaced by d_r. To the first order in $V/\omega r^2$ and $(V/\omega r)\,\partial_r$,

$$\begin{gathered}u_r = -(i/\omega\rho)\,[1 - (iV/\omega r)\,\partial_r + (iV/\omega r^2)]\,\partial_r p,\\ u_\phi = -(i/\omega\rho)\,[1 - (iV/\omega r)\,\partial_r - (iV/\omega r^2)]\,(imp/r).\end{gathered} \tag{7}$$

This is substituted in the first line of equation (6), yielding, to the first order in the velocity,

$$\begin{aligned}\kappa^2 p + p'' + p'/r - m^2 p/r^2 &= (iV/\omega r)\,[p''' - p''/r + (p'/r^2)(1-m^2) - \kappa^2 p'] \\ &\equiv i^m V f_m(\kappa r)\, e^{im\phi - i\omega t}, \\ \kappa &= \omega/c, \end{aligned} \tag{8}$$

where the prime denotes ∂_r. To first-order accuracy, the solution of equation (8) may be written in the form

$$p = p_0 + V p_1, \tag{9}$$

where p_0 is the conventional solution for media at rest. Thus

$$p_0 = \sum_{m=-\infty}^{\infty} i^m A_{m0} Z_m(\kappa r)\, \mathrm{e}^{im\phi - i\omega t}, \tag{10}$$

where Z_m is a solution of the Bessel equation of order m; A_{m0} is an arbitrary constant, determined by boundary conditions. The expression in brackets in equation (8) is already of first order in the velocity; therefore from equation (9) one may substitute only p_0 in this expression. Consequently it is seen that equation (8) becomes an inhomogeneous equation, and equation (10) is its homogeneous solution. Rewriting equation (8) as

$$\nabla^2 p + \kappa^2 p = i^m V f_m(\kappa r)\, \mathrm{e}^{im\phi - i\omega t}, \tag{11}$$

makes it clear that the particular solution of equation (8) constitutes the velocity correction Vp_1 in equation (9), and this can be found by solving the inhomogeneous equation (11). This is done by using the free-space Green function integral (for the theory see, for example, Morse and Feshbach [6] and Morse and Ingard [5]). The two-dimensional free-space Green function is defined by

$$\begin{aligned}(\nabla^2 + \kappa^2) G &= -\delta(\mathbf{r} - \mathbf{r}_0), \\ \delta(\mathbf{r} - \mathbf{r}_0) &= \delta(x - x_0)\,\delta(y - y_0),\end{aligned} \tag{12}$$

and it is readily shown that

$$G = (i/4)\, H_0(\kappa|\mathbf{r} - \mathbf{r}_0|), \tag{13}$$

where H_0 is the Hankel function of the first kind, of order zero. Therefore the particular solution of equation (11) is, for the mth mode,

$$p_1(\kappa r) = -\int_S G(\mathbf{r}, \mathbf{r}_0) f_m(\kappa r_0)\, i^m\, \mathrm{e}^{im\phi_0 - i\omega t}\, \mathrm{d}S(\mathbf{r}_0), \tag{14}$$

integrated over the plane S where p_0 exists. For the subsequent problem of a cylindrical cavity of radius a we are interested in $p_1(\kappa a)$. Using the addition theorem for cylindrical functions,

$$G = (i/4) \sum_{m=-\infty}^{\infty} H_m(\kappa r) J_m(\kappa r_0)\, \mathrm{e}^{im(\phi - \phi_0)}, \qquad r > r_0, \tag{15}$$

and exploiting the orthogonality of $\mathrm{e}^{im\phi_0}$, one has

$$p_1(\kappa a) = (2\pi/4i)\, i^m\, \mathrm{e}^{-i\omega t + im\phi} H_m(\kappa a) \int_a^{\infty} J_m(\kappa r_0) f_m(\kappa r_0)\, r_0\, \mathrm{d}r_0, \tag{16}$$

for the mth mode. The last integral (which is related to the Hankel transform) is easily computerized. As $r \approx \infty$ (i.e., in the far field) $p_1(\kappa r)$ vanishes faster than p_0; this is expected in view of equation (3), as the motion of the medium decreases with distance from the origin. For subsequent considerations equation (16) is of importance for evaluating the scattering coefficients.

For arbitrary distances one has to supplement equation (15) with the expansion for $r < r_0$, which is obtained by exchanging r and r_0; hence

$$p_1(\kappa r) = (2\pi/4i)\,\mathrm{e}^{-i\omega t + im\phi}[H_m(\kappa r)\,I_> + J_m(\kappa r)\,I_<],$$

$$I_> = \int_r^\infty J_m(\kappa r_0) f_m(\kappa r_0)\, r_0\,\mathrm{d}r_0,$$

$$I_< = \int_a^r H_m(\kappa r_0) f_m(\kappa r_0)\, r_0\,\mathrm{d}r_0. \tag{17}$$

In spherical coordinates the second line of equation (3) is assumed. As only the case of an axisymmetrical pressure is of interest here (i.e., if z is the polar axis, the problem is independent of the azimuthal coordinate ϕ defined in a plane perpendicular to z), the problem depends only on r, the distance from the origin, and θ, the polar angle. From equations (2) and (5) the analogue of equation (6) can be obtained:

$$\begin{aligned} -i\omega c^{-2}[1 + (iV/\omega r^2)\,\partial_r]p + \rho[(\partial_r + 2/r)\,u_r + (r\sin\theta)^{-1}\,\partial_\theta(\sin\theta u_\theta)] &= 0,\\ \partial_r p - i\omega\rho[1 + (iV/\omega r^2)\,\partial_r - (2iV/\omega r^3)]\,u_r &= 0,\\ r^{-1}\,\partial_\theta p - i\omega\rho[1 + (iV/\omega r^2)\,\partial_r + (iV/\omega r^3)\,u_\theta &= 0. \end{aligned} \tag{18}$$

In a manner similar to that used in connection with equation (7), to the first order in the velocity, u_r and u_θ can be isolated and substituted into the first line of equation (18). Using the fact that the pressure has a period π in θ then gives

$$(\sin\theta)^{-1}\,\partial_\theta(\sin\theta\,\partial_\theta)p = -n(n+1)p, \tag{19}$$

for the nth mode. This result follows from the separation of variables of the wave equation in spherical coordinates. Thus the analogue of equation (8) is found to be

$$\begin{aligned} \kappa^2 p + p'' + (2/r)p' - n(n+1)p/r^2 &= (iV/\omega r^2)\{p''' - 2p''/r + [2 - n(n+1)]p'/r^2\}\\ &\equiv i^n(2n+1)\,Vf_n(\kappa r)\,P_n(\cos\theta)\,\mathrm{e}^{-i\omega t}, \end{aligned} \tag{20}$$

where again the prime denotes ∂_r and $P_n(\cos\theta)$ is the nth order Legendre polynomial. There is no danger of confusing f_n with f_m of equation (8), because of the index. Again $p = p_0 + VP_1$ as in equation (9), but now V and p_1 have different dimensions. The velocity-independent part, by analogy with equation (10), is,

$$p_0 = \sum_{n=0}^{\infty} i^n(2n+1)\,A_{n0}\,z_n(\kappa r)\,P_n(\cos\theta)\,\mathrm{e}^{-i\omega t}, \tag{21}$$

where z_n is the spherical Bessel function. Again we consider the solution of the inhomogeneous wave equation (20), considering f_n to be a known function, by substituting equation (21) in the expression in braces in equation (20). The analogue of equation (12) now relates to $\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)$ which is integrated over $\mathrm{d}v_0 = \mathrm{d}x_0\,\mathrm{d}y_0\,\mathrm{d}z_0$. Corresponding to equation (13) we now have

$$G(\mathbf{r}, \mathbf{r}_0) = (i\kappa/4\pi)\,h_0(\kappa|\mathbf{r} - \mathbf{r}_0|), \tag{22}$$

where h_0 is the zero-order spherical Hankel function of the first kind. The addition theorem for spherical wave functions (e.g., see Morse and Feshbach [6], p. 887) prescribes

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}_0) = (i\kappa/4\pi) \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1)\,[(n-m)!/(n+m)!]\,\mathrm{e}^{im(\phi-\phi_0)}\,\cdot\\ \cdot P_n^m(\cos\theta)\,P_n^m(\cos\theta_0) \begin{cases} j_n(\kappa r)\,h_n(\kappa r_0);\ r \leqslant r_0,\\ j_n(\kappa r_0)\,h_n(\kappa r);\ r \geqslant r_0, \end{cases} \end{aligned} \tag{23}$$

where j_n is the non-singular spherical Bessel function and ϕ and ϕ_0 are azimuthal angles. The particular solution is

$$p_1(\kappa r) = \mathrm{e}^{-i\omega t} \int_v G(\mathbf{r}, \mathbf{r}_0) f_n(\kappa r_0) P_n(\cos\theta_0)\, i^n(2n+1)\, \mathrm{d}v(r_0), \tag{24}$$

integrated over the volume where p_0 exists. For a spherical cavity of radius a one obtains, similar to equation (16),

$$p_1(\kappa a) = i\kappa\, \mathrm{e}^{-i\omega t}\, i^n(2n+1) P_n(\cos\theta)\, h_n(\kappa a) \int_a^\infty j_n(\kappa r_0) f_n(\kappa r_0)\, r_0^2\, \mathrm{d}r_0, \tag{25}$$

and the analogue of equation (17) follows by inspection.

This completes the derivation of the first-order velocity correction for sound waves in radially flowing media.

3. SCATTERING BY RADIALLY EXPANDING CAVITIES

When a circular cylinder or a sphere expands radially, the surrounding medium is also displaced; therefore the theory of the previous section is applicable. The effect vanishes in the far field, where the observer is usually situated, but is significant at the surface of the scatterer, where the boundary conditions are applied, determining the scattering coefficients.

Although the radial velocity of the surface is varying with time, it is treated here as a constant; this is plausible for short periods of observation. The position of the surface is described by

$$r = a + vt, \tag{26}$$

where a is the radius and v is the velocity in the vicinity of $t = 0$.

The incident wave, p_i, is chosen to be a monochromatic plane pressure wave at large distances from the scatterer. Its velocity-independent part is therefore given by

$$p_{i0} = \mathrm{e}^{ikx - i\omega t} = \sum_{m=-\infty}^{\infty} i^m J_m(\kappa r)\, \mathrm{e}^{im\phi - i\omega t}, \qquad x = r\cos\phi,$$

$$p_{i0} = \mathrm{e}^{ikz - i\omega t} = \sum_{n=0}^{\infty} i^n(2n+1) j_n(\kappa r) P_n(\cos\theta), \qquad z = r\cos\theta, \tag{27}$$

for, respectively, the cylindrical case independent of the axial coordinate z, and for the spherical case independent of the azimuthal angle ϕ. J_m and j_n are the non-singular cylindrical and spherical functions, respectively. In the first and second line of equation (27), r refers to the radial coordinate in the relevant coordinate system.

In the vicinity of the scatterer a velocity correction term must be added. At $r = a$, $p_{i1}(\kappa a)$ for each mode in cylindrical and spherical coordinates is given by equation (16) or by equation (24), respectively. The integrals are evaluated by using equation (27) as the p_0 in f_m or f_n as explained in the previous section. In order to introduce explicitly the dependence on modes, one can write,

$$p_i = p_{i0} + Vp_{i1},$$

$$p_{i1}(\kappa a) = \sum_{m=-\infty}^{\infty} i^m\, Q_{i,m}(\kappa a)\, \mathrm{e}^{im\phi - i\omega t},$$

$$p_{i1}(\kappa a) = \sum_{m=0}^{\infty} i^n(2n+1)\, q_{i,n}(\kappa a) P_n(\cos\theta)\, \mathrm{e}^{-i\omega t}, \tag{28}$$

for the cylindrical and spherical cases, respectively. Since Vp_{i1} is already of first order in the velocity, the boundary conditions are applied to it at $r = a$, the time dependence in equation (26) being neglected.

One cannot assume that the scattered wave is monochromatic. Such a postulate leads to coefficients A_m in equation (10), or A_n in equation (21), which are time dependent; hence the result is not a proper solution of the Helmholtz wave equation. In general this would mean that one has to solve the problem by allowing an arbitrary time dependence: i.e., the hyperbolic wave equation must be solved, rather than the Helmholtz equation. However, to the first order in the velocity it is feasible to assign a different frequency to each mode, and get a solution to the problem. This has been done previously for the analogous problems in electrodynamics [2]. The following *ansatz* is made:

$$p_s = \sum_{n=-\infty}^{\infty} i^m A_m[H_m(\kappa_m r) + VQ_{s,m}(\kappa r)]\,\mathrm{e}^{im\phi - i\omega t - iB_m Vt},$$

$$p_s = \sum_{m=0}^{\infty} i^n(2n+1)A_n[h_n(\kappa_n r) + Vq_{s,n}(\kappa r)]\,P_n(\cos\theta)\,\mathrm{e}^{-i\omega t - ib_n Vt},$$

$$\kappa_m = (\omega + VB_m)/c, \qquad \kappa_n = (\omega + Vb_n)/c, \tag{29}$$

such that each mode is a solution of the Helmholtz equation, and for each mode there is a different Doppler shift. The structure of $Q_{s,m}$ and $q_{s,n}$ is similar to that shown in equation (28), hinging on the solution of the inhomogeneous wave equation as exemplified by equation (17). The choice of H_m and h_n together with the time factor $\mathrm{e}^{-i\omega t}$ satisfies the usual radiation condition, since at large distances the velocity effects, Q and q, vanish.

The boundary condition is

$$p_i + p_s = 0|_{r=a+v(a)t}, \tag{30}$$

where $v(a)$ is given by equation (3) when $r = a$. This is applied to equations (27)–(29). The expressions are expanded in Taylor series, where first-order velocity effects only are retained, and it is assumed that the period of observation is small, such that $B_m Vt$ and $b_n Vt$ are small. For terms which are already of first order in the velocity we use

$$A_{m0} = -J_m(\kappa a)/H_m(\kappa a),$$
$$A_{n0} = -j_n(\kappa a)/h_n(\kappa a), \tag{31}$$

which are the conditions for pressure vanishing at $r = a$ for static scatterers. Equating to zero all time dependent and time constant terms yields

$$B_m = (i\kappa/a)\,[J_m'(\kappa a)/J_m(\kappa a) - H_m'(\kappa a)/H_m(\kappa a)],$$
$$b_n = (i\kappa/a^2)\,[j_n'(\kappa a)/j_n(ka) - h_n'(\kappa a)/h_n(\kappa a)], \tag{32}$$

and

$$A_m = A_{m0} - (V/H_m(\kappa a))\,[Q_{i,m}(\kappa a) + A_{m0}\,Q_{s,m}(\kappa a) + (aB_m A_{m0}/c)\,H_m'(\kappa a)],$$
$$A_n = A_{n0} - (V/h_n(\kappa a))\,[q_{i,n}(\kappa a) + A_{n0}\,q_{s,n}(\kappa a) + (ab_n A_{n0}/c)\,h_n'(\kappa a), \tag{33}$$

where the prime indicates differentiation with respect to the argument.

4. DISCUSSION

The fact that equations (32) and (33) are constants justifies *a posteriori* the construction (29). Consequently, within the accuracy of first-order velocity effects, and in the vicinity of $t = 0$, the scattered field is known. The results can be evaluated for various instants of time, for each case $t = 0$ being redefined. Thus the dependence on time may be established, which can be used to get characteristic signatures for various processes.

Usually the observer is situated in the far field. This simplifies equation (29) since $Q_{s,m}$ and $q_{s,n}$ are negligible and

$$i^m H_m(\kappa_m r) \approx (2/i\kappa_m r)^{1/2}\,\mathrm{e}^{i\kappa_m r},$$
$$i^n h_n(\kappa_m r) \approx (i\kappa_n r)^{-1}\,\mathrm{e}^{i\kappa_m r}, \tag{34}$$

so that we do not have to evaluate Hankel functions of complex argument. Together with the time factor $e^{-i\omega_m t}$, equation (34) describes the effect of retardation. Thus the time $t=0$ in the vicinity of the scatterer corresponds to $t=r/c$ at a distance r. Hence at a large distance r at time $t=r/c$,

$$P_s \sim (2/i\pi\kappa r)^{1/2} \sum_m (1 - VB_m/2\kappa c) A_m e^{im\phi}. \tag{35}$$

Now, if we define the scattering cross-section A in the usual way,

$$\frac{\kappa A}{4} = \frac{1}{2\pi} \int_0^{2\pi} d\phi g g^*, \tag{36}$$

where g denotes the sum in equation (35), then

$$\frac{\kappa A}{4} = \sum_m A_m A_m^* (1 - VB_m/\kappa c). \tag{37}$$

Thus in the scattering cross section the dependence on the distance of the observer from the scatterer does not enter, and the first-order velocity effects are included. A similar argument applies to the spherical case.

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