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COMMENTS ON "SCATTERING BY TIME VARYING OBSTACLES"

In a recent article [1], Censor attempts to solve the problem of the scattering of a plane wave (of angular frequency ω) by an obstacle whose surface deforms harmonically with angular frequency Ω. The general approach of the paper is to solve by means of a perturbation method the *linear* wave equation with time dependent boundary conditions. The result is that in addition to the ordinary rigid body scattering, waves at frequencies $\omega_\pm = \omega \pm \Omega$ are produced. Censor has no doubt correctly solved the problem of the linear wave equation with time dependent boundary conditions but I do not believe that he has correctly solved the problem of scattering by time varying obstacles. I believe that this problem is inherently non-linear and cannot be solved by means of the linear wave equation. Consider the following arguments.

(1) It is well known [2, 3] that when sound fields of frequencies ω and Ω are present in a fluid, the non-linear terms in the wave equation act as volume sources of sound at frequencies $\omega \pm \Omega$. At least in the case of the "rigid" scatterer this non-linear "volume" effect would completely overwhelm the surface effect described by Censor except possibly within a fraction of a wavelength of the scatterer. This is due to the fact that although both effects are second order, the non-linear "volume" effect is cumulative with distance, while Censor's effect is not. Therefore, although sound at frequencies $\omega_\pm$ would undoubtedly be present in the fluid, for the most part, the presence of these frequencies would not be due to the mechanism considered by Censor. It is important to recognize that the $\omega_\pm$ generated by the non-linear volume sources cannot be decreased relative to the $\omega_\pm$ due to Censor's effect by decreasing either of the Mach numbers (viz., the Mach number of the incident plane wave and the Mach number associated with the motion of the scattering surface) because both effects depend on the Mach numbers in the same way. There is no way to linearize the wave equation by decreasing the Mach number without at the same time eliminating Censor's effect.

(2) Acoustic parameters may be considered in either of two representations, Eulerian or Lagrangian [4]. In the Eulerian representation a parameter $\theta(\mathbf{r},t)$—which can be either a vector or a scalar quantity—represents the value of that parameter at the point $\mathbf{r}$, and at time t. In the Lagrangian representation a parameter $\theta^{\mathrm{L}}(\mathbf{r},t)$ (superscript L denotes Lagrangian coordinates) represents the value of that parameter at time t for the "fluid particle" that was located at point $\mathbf{r}$ in the undisturbed fluid. That is, with Eulerian coordinates one observes the fluid at a point fixed in space, but with Lagrangian coordinates at a point that moves with the local fluid velocity. To second order, $\theta^{\mathrm{L}}(\mathbf{r},t)$ is related to $\theta(\mathbf{r},t)$ by the equation

$$\theta^{\mathrm{L}}(\mathbf{r},t) = \theta(\mathbf{r},t) + [\boldsymbol{\xi}(\mathbf{r},t)\cdot\nabla]\,\theta(\mathbf{r},t), \tag{1}$$

where $\boldsymbol{\xi}(\mathbf{r},t)$ is the displacement of the particle originally located at point $\mathbf{r}$.

The linear wave equation is obtained from the exact wave equation by neglecting all terms which are second order in the Mach number. From equation (1) it can be seen that Lagrangian and Eulerian quantities differ only by terms which are second order in the Mach number. It is therefore inconsistent to use the linear wave equation and still retain any distinction between Lagrangian and Eulerian coordinates. When the linear wave equation is used, any differences between the results obtained in the Lagrangian representation and the results obtained in the Eulerian representation are physically meaningless since differences must be second order and many other (possibly more significant) second-order effects have already been neglected. Censor chooses to interpret his dependent variables as Eulerian quantities.

It is the retention of the second-order terms which result from applying the boundary condition in the Eulerian representation which produces the effect he predicts. If he had interpreted his parameters in the *Lagrangian* sense he would have obtained *no effect*. Censor's model for scattering by a vibrating obstacle therefore cannot be physically correct.

To see that Censor's approach yields no sum or difference frequencies in the Lagrangian representation, consider the case of scattering from a vibrating "rigid" cylinder. In Lagrangian coordinates the boundary condition is

$$u_r^L(a, \theta, t) = a\Omega\varepsilon\cos(\Omega t), \tag{2}$$

where u_r^L is the total radial velocity (incident + scattered + radiated) and a is the radius of the cylinder. This is true because $r = a$ defines the surface of the cylinder in Lagrangian coordinates at all times no matter what the motion of the surface may be. Let $\phi(r,\theta,t)$ be the total velocity potential. If $\phi(r,\theta,t)$ is written in the form

$$\phi(r,\theta,t) = \phi_\omega(r,\theta)\,e^{i\omega t} + \phi_\Omega(r,\theta)\,e^{i\Omega t}, \tag{3}$$

it is found that $\phi_\omega(r,\theta)$ and $\phi_\Omega(r,\theta)$ satisfy the Helmholtz equation with wave numbers $k_\omega = \omega/c$ and $k_\Omega = \Omega/c$, respectively, and the boundary conditions

$$\left.\frac{\partial\phi_\omega(r,\theta)}{\partial r}\right|_{r=a} = 0, \tag{4}$$

$$\left.\frac{\partial\phi_\Omega(r,\theta)}{\partial r}\right|_{r=a} = -a\Omega\varepsilon. \tag{5}$$

Thus the problem decomposes into the *separate* and *independent* problems of rigid body scattering from a cylinder at frequency ω and radiation from a cylinder at frequency Ω. No sum or difference frequencies are produced in the Lagrangian representation when the linear wave equation is used. Such sum and difference frequencies can come only from non-linear terms in the wave equation.

Censor's analysis includes some second-order terms (i.e., those which distinguish between the Eulerian and Lagrangian boundary conditions) but excludes other (i.e., the second-order terms which appear in the wave equation in either representation). The implicit assumption that the linear wave equation is correct to second order in the Eulerian representation is not justifiable. Censor's analysis predicts effects which are second order; he must, therefore, retain all second-order terms or justify their neglect.

(3) Although it is not immediately apparent, the arguments presented in (1) and (2) above also apply to Censor's example involving transverse waves on a stretched membrane. The transverse and longitudinal motions are indeed decoupled correct to first order but, in general, correct to second order, they must be coupled. The equation governing the transverse displacement ξ_z of a membrane of surface density σ and tension T is [5]

$$\sigma\,\partial^2\xi_z/\partial t^2 = T[\partial/\partial x^2 + \partial/\partial y^2]\,\xi_z, \tag{6}$$

which is seemingly independent of the longitudinal displacements ξ_x and ξ_y. Closer examination, however, reveals that both the density σ and the tension T must depend on ξ_x and ξ_y. Any local stretching or compression of the membrane must change the local density and tension. For example, if σ_0 is the density when ξ_x and ξ_y are zero then

$$\sigma = \sigma_0(1 + \partial\xi_x/\partial x + \partial\xi_y/\partial y)^{-1}, \tag{7}$$

since $\nabla\cdot\xi$ is the fractional change in the area of a surface element of the membrane. From equations (6) and (7) it is evident that the equations governing the motion of a membrane which is excited both longitudinally and transversely are non-linear.

Though I believe Censor's analysis to be invalid for mechanical waves, I believe that his analysis would be valid for electromagnetic waves. In particular, he has correctly solved the problem of scattering of an electromagnetic wave by a vibrating conductor. The electromagnetic wave equation (for a vacuum) is known to be linear in any Galilean reference frame. The boundary condition *must* be interpreted in the Eulerian sense since Lagrangian coordinates constitute an accelerated reference frame.

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AUTHOR'S REPLY

In the paper [1] referred to by Rogers [2] it is assumed that changes in the medium are small, such that if ω, Ω frequencies are present anywhere in the fluid, no $\omega \pm \Omega$ frequencies are produced. Although it is well known that non-linear behaviour produces the parametric effect (generation of $\omega \pm \Omega$ and other frequencies), the aim of reference [1] was to show that new frequencies are produced due to the time varying boundary conditions, even if the volume effects are negligible.

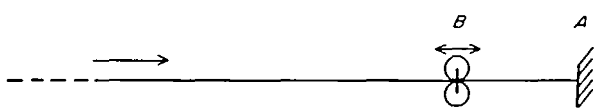

Figure 1. Sketch of string with sliding obstacle (B).

Starting with Rogers' [2] third argument, let us examine the problem of the membrane, or rather its limiting case of a string, as described in Figure 1. The string is anchored at A and "infinity"; the incident wave comes from the left. The obstacle B slides on the string without friction (symbolized by the rollers), and therefore for small amplitude transverse waves, changes in tension and density are negligible. The analysis [3, 4] shows that Doppler effects (changes in frequency and wavelength) are produced. For this case and the corresponding membrane problems in reference [1] the use of the linear wave equations is legitimate, and the production of new frequencies is effected by the time dependent boundary conditions. Neglecting the terms that depend on the motion of the boundary amounts to denying the possibility of a Doppler effect.

This should answer one of the objections raised by Rogers [2]: namely, that non-linear volume effects and time dependent boundary conditions as in reference [1] cannot be considered independently.