

Generalized Doppler Effect in Time-varying Media

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ABSTRACT: *Doppler effects for time-varying inhomogeneous media are considered. The ray path is represented in terms of a two-parameter function, this takes into account its time evolution. Time dilation along the path is discussed. This provides, in a systematic way, an alternative derivation to a heuristic expression previously given by Gill. The phase function is integrated along the ray path and yields the various Doppler effects in a straightforward way. It is shown that the generalized Doppler effect is produced by the relative motion of source and receiver with respect to the medium, and the cumulative rate of change of frequency and propagation vector along the path.*

I. Introduction

Formulation of the Doppler effect in a time-invariant, homogeneous medium is already well established and needs no further comment. For an inhomogeneous non-absorbing time-invariant medium whose properties are spatially slowly varying the existing derivations are likewise satisfactory. See, for example, Guier and Weiffenbach (1), Bennet (2)–(4), Kelso (5), Rawer and Suchy (6), Gill (7) and a recent paper by Fante (8).

The problem of ray propagation in time-varying inhomogeneous media has been considered by several authors, see (2)–(8) and (9). However, in all derivations, as far as we are aware, when Doppler effects are considered, the frequency is taken as a constant in time and space. Essentially, the derivations hinge on the concept of rate of change of the optical path length. It is not clear how these results can be extended to the general case where both the wave vector \mathbf{k} and the (angular) frequency ω are functions of position and time. Fante (8) derives the Doppler effect from the phase, but neglects the changes in the frequency.

In the following, we develop a formula for the time dilation along the path, which serves to replace the heuristic result given by Gill (7). However, the interpretation of the Doppler effect in terms of the time dilation is incomplete. To overcome this difficulty we use an alternative approach, deriving the Doppler effect from the phase function appropriate to time-varying inhomogeneous media, slowly varying in space and time.

II. The Ray Path and Time Dilation

We are dealing with a time-varying inhomogeneous medium, making the usual assumptions that waves propagate, having local plane wave character, and that ray paths can be identified.

Let us follow some point on the wave. This signal has been emitted at time t_0 and arrives at the receiver at time t , after a time interval τ , i.e.

$$t_0 = t - \tau. \tag{1}$$

The path of the signal can be specified by a function of two parameters,

$$\mathbf{r} = \mathbf{R}(\tau, t). \tag{2}$$

This describes a family of curves in space. The parameter t tags the ray path for a signal which arrived at the receiver at time t . Along this specific ray path, the parameter τ locates the point where the signal was present at time $t - \tau$. Note that τ, t are independent variables. For different t , signals traverse different paths, because of the time-dependent changes in the medium.

In a time-invariant medium ω can be found by measuring

$$\Delta\phi = \phi(\mathbf{r}, t + dt) - \phi(\mathbf{r}, t)$$

at a fixed position \mathbf{r} , and in the limit $\omega = -\lim_{\Delta t \rightarrow 0}(\Delta\phi/\Delta t)$. In the time-varying case the frequency changes spatially and temporally and $\omega(\mathbf{r}, t)$ cannot be ascertained from a local measurement and the complete information along the integration path is necessary, between events (\mathbf{r}_0, t_0) to the neighboring events (\mathbf{r}, t) , $(\mathbf{r}, t + dt)$ in four-space. If we simply measure the phase at a fixed point \mathbf{r} at times t and $t + dt$, we actually perform $\Delta\phi^* = \phi_1(\mathbf{r}, t + \Delta t) - \phi_2(\mathbf{r}, t)$, where ϕ_1 and ϕ_2 are integrated along different ray paths. Hence the observed frequency $\omega^* = -\lim_{\Delta t \rightarrow 0}(\Delta\phi^*/\Delta t)$ is different from the "intrinsic" frequency $\omega(\mathbf{r}, t)$, and the discrepancy is the Doppler effect due to the time-varying medium. Thus, as a function of t , the family $\mathbf{r} = \mathbf{R}(\tau, t)$ describes the time evolution of the path. For a time-invariant medium the path shape is independent of the time of arrival t , and (2) reduces to $\mathbf{r} = \mathbf{R}(\tau)$. A signal starting at time t_0 will take a time τ_0 to reach the receiver, hence $\mathbf{r} = \mathbf{R}(\tau_0, t)$ describes the source point and $\mathbf{r} = \mathbf{R}(0, t)$ denotes the observer. It follows that τ is monotonically increasing in the direction opposite to that of the signal. Obviously, the path can be tagged by the starting time t_0 , this merely exchanges the roles of transmitter and receiver ends.

The path length will be defined as

$$p(\tau, t) = \int_0^\tau C(\mathbf{r}', t') d\tau', \tag{3}$$

where $C(\mathbf{r}', t')$ is the speed of propagation in the time-varying medium, and the primes denote variables of integration. A signal arriving at t will traverse a path of length $p(\tau, t)$ starting from a point tagged by τ . At any point $\mathbf{r}' = \mathbf{R}(\tau', t)$ along the path, the remaining "time of flight" is τ' . The function $C(\mathbf{r}', t')$ must be evaluated at the appropriate time $t' = t - \tau'$, hence by substitution,

$$C(\mathbf{r}', t') = C(\mathbf{R}[\tau', t], t - \tau') \equiv \mathcal{C}(\tau', t). \tag{4}$$

Consequently, by knowing the path and $C(\mathbf{r}', t')$ on it $p(\tau, t)$ can be found by integration of (3).

Let us first consider the differential of (3),

$$dp = \frac{\partial p}{\partial \tau} d\tau + \frac{\partial p}{\partial t} dt, \tag{5}$$

which is the total change of path length due to independent changes in the parameters τ, t . Holding t fixed, which means that we are on one member of the family, (1) and (3) yield

$$dp|_t = \frac{\partial p}{\partial \tau} d\tau = \mathcal{C}(\tau, t) d\tau = -C(\mathbf{r}_0, t_0) dt_0, \tag{6}$$

where $\mathbf{r}_0 = \mathbf{R}(\tau, t)$. Physically Eq. (6) describes the path difference between two signals, both arriving at the receiver along the same path, i.e. simultaneously. One signal is emitted at time t_0 and travels for a duration τ , the other started *earlier* by an amount $-dt_0$ and travels a longer time $\tau + d\tau$, such that $d\tau = -dt_0$. At time t_0 the signals overlap and move together along the path. If we keep τ fixed, then

$$dp|_t = \frac{\partial p}{\partial t} dt = dt \int_0^\tau \frac{\partial}{\partial t} \mathcal{C}(\tau', t) d\tau'. \tag{7}$$

This now describes two signals arriving at the time difference dt and having the same time of flight τ . The signals move along neighboring paths and $dp|_t$ is the path length difference due to the changes in the medium. In the present case $dt = dt_0$, since τ is fixed, this means that the signals were emitted at a time interval dt_0 equal to dt , the time lapse at arrival.

We are interested in the time dilation along the path. If two signals are emitted with a time lapse dt^* from the same transmitter, and arrive with a time lapse dt at the receiver, how is the ratio dt^*/dt related to the properties of the medium? To answer this question, consider two signals emitted simultaneously but at neighboring points. This means $dt_0 = 0$ and from (1) $d\tau = dt$. Equation (5) therefore becomes

$$dp = \left[C(\mathbf{r}_0, t_0) + \frac{\partial p}{\partial t} \right] dt. \tag{8}$$

The path difference dp will be covered by a signal moving at a speed $C(\mathbf{r}_0, t_0)$ during a time interval dt^* . This is the same as saying that the signals have been emitted at adjacent points on neighboring paths (which coalesce as $dt \rightarrow 0$, defining the position of the transmitter) at a time lapse dt^* . Hence (8) can be rewritten as

$$\frac{dt^*}{dt} = 1 + \frac{1}{C(\mathbf{r}_0, t_0)} \int_0^\tau \frac{\partial}{\partial t} \mathcal{C}(\tau', t) d\tau', \tag{9}$$

describing the time dilation along the ray path. The same formula can be derived from (5) without the condition $dt_0 = 0$ if we define

$$dp - C(\mathbf{r}_0, t_0) dt_0 \equiv C(\mathbf{r}_0, t_0) dt^*.$$

The formula given by Gill (7) for the time of flight for the case of the time-invariant medium is

$$t - t_0 = \int_p \frac{\mu(s)}{c} ds, \tag{10}$$

where $\mu(s)$ is the index of refraction along the path, c is the reference speed and the integral is taken along the path. By differentiation (for fixed transmitter and receiver, say), the time dilation ratio according to Gill is

$$\frac{dt}{dt_0} = 1 + \frac{1}{c} \int_p \frac{\partial}{\partial t_0} \mu(s) ds. \tag{11}$$

For the case of a time-invariant medium the integral vanishes. However, for the time-varying medium (11) fails to show explicitly how $\mu(s)$ is a function of t_0 or t . We believe that (9), with the defined parametrization of the ray path, is a more systematic result and allows for a clear interpretation of the integral.

The heuristic interpretation that the ratio of the time increments is inversely proportional to the corresponding frequencies is neither used nor needed here. It is believed that the Doppler effects should emerge from well-defined operations on a properly constructed phase function.

III. The Phase Function and Doppler Effects

In a slowly varying medium the phase function will be given by

$$\phi = \int_L (\mathbf{k} \cdot d\mathbf{r}' - \omega dt') \tag{12}$$

[see (8) for additional details and references]. It is understood that the real (or imaginary) part of $\exp(i\phi)$ describes the field and is physically measurable. Here $\mathbf{k} = \mathbf{k}(\mathbf{r}', t')$, $\omega = \omega(\mathbf{r}', t')$ depend on space and time and L is any path of integration in four-space between events (r_0, t_0) and (\mathbf{r}, t) . From this follows

$$\frac{\partial \phi}{\partial t} = -\omega(\mathbf{r}, t), \quad \nabla \phi = \mathbf{k}(\mathbf{r}, t), \tag{13}$$

and with proper assumptions on the differentiability of ϕ we get Whitham's formula (10)

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla \omega = 0, \tag{14}$$

which includes the Sommerfeld–Runge law

$$\nabla \times \mathbf{k} = 0, \tag{15}$$

in an obvious way.

In order to obtain the Doppler effects, the path L must be specified as the actual ray path, as given by Eqs. (1) and (2). With the proper

substitutions as in Eq. (4), we define

$$\left. \begin{aligned} \mathbf{k}(\mathbf{r}', t') &\equiv \mathbf{K}(\tau', t), \\ \omega(\mathbf{r}', t') &\equiv W(\tau', t). \end{aligned} \right\} \quad (16)$$

Since the integration is along a ray path for which the tag t is a constant, we have $dt' = -d\tau'$ and $d\mathbf{r}' = (\partial\mathbf{R}/\partial\tau') d\tau'$, yielding

$$\Phi(\tau, t) = \int_{\tau}^0 \left[\mathbf{K}(\tau', t) \cdot \frac{\partial\mathbf{R}}{\partial\tau'} d\tau' + W(\tau', t) d\tau' \right]. \quad (17)$$

The increment of the phase due to increments in τ and t is

$$\begin{aligned} d\Phi(\tau, t) &= \frac{\partial\Phi}{\partial\tau} d\tau + \frac{\partial\Phi}{\partial t} dt \\ &= - \left[\mathbf{K}(\tau, t) \cdot \frac{\partial\mathbf{R}(\tau, t)}{\partial\tau} + W(\tau, t) \right] d\tau \\ &\quad + dt \int_{\tau}^0 \frac{\partial}{\partial t} \left[\mathbf{K}(\tau', t) \cdot \frac{\partial\mathbf{R}}{\partial\tau'} + W(\tau', t) \right] d\tau'. \end{aligned} \quad (18)$$

This describes the total change of phase between two neighboring paths, one starting at t_0 at $\mathbf{r}_0 = \mathbf{R}(\tau, t)$ and ending at t at $\mathbf{r} = \mathbf{R}(0, t)$, the other starting at $\mathbf{r}_0 + d\mathbf{r}_0 = \mathbf{R}(\tau + d\tau, t + dt)$, ending at an adjacent point $\mathbf{R}(0, t + dt)$. However, to have a concept of a transmitter and a receiver, we must consider adjacent points on neighboring paths also at the source end of the paths. This is similar to considerations made above in connection with the time dilation. Substituting $d\tau = dt - dt_0$ and identifying

$$\mathbf{K}(\tau, t) \cdot \frac{\partial\mathbf{R}(\tau, t)}{\partial\tau} d\tau = \mathbf{k}(\mathbf{r}_0, t_0) d\mathbf{r}_0 \quad \text{and} \quad W(\tau, t) = \omega(\mathbf{r}_0, t_0),$$

(18) becomes

$$\left. \begin{aligned} d\Phi + \mathbf{k}(\mathbf{r}_0, t_0) \cdot d\mathbf{r}_0 - \omega(\mathbf{r}_0, t_0) dt_0 &= -\omega(\mathbf{r}_0, t_0) dt, \\ -dt \int_0^{\tau} \frac{\partial}{\partial t} \left[\mathbf{K}(\tau', t) \cdot \frac{\partial\mathbf{R}}{\partial\tau'} + W(\tau', t) \right] d\tau' &\equiv d\Phi^*. \end{aligned} \right\} \quad (19)$$

In the limit this corresponds to the difference in phase between transmitter and receiver, with the phase integrated along neighbouring paths. The observed frequency, measured at the receiver is $\omega^* = -d\Phi^*/dt$, hence we finally have

$$\omega^* = \omega(\mathbf{r}_0, t_0) + \int_0^{\tau} \frac{\partial}{\partial t} \left[\mathbf{K}(\tau', t) \cdot \frac{\partial\mathbf{R}}{\partial\tau'} + W(\tau', t) \right] d\tau', \quad (20)$$

which defines the Doppler effect due to the changes of \mathbf{k} and ω as caused by the time-varying inhomogeneous medium.

In this result the effects of changes of the propagation vector and the frequency are both included.

Equation (20) as it stands is applicable to investigation of time-changing media, e.g. the ionosphere. However, if the Doppler effects due to moving receiver and transmitter are to be included, we replace ω^* by $\omega^* - \mathbf{k}(\mathbf{r}, t) \cdot \mathbf{V}(\mathbf{r}, t)$, and $\omega(\mathbf{r}_0, t_0)$ by $\omega(\mathbf{r}_0, t_0) - \mathbf{k}(\mathbf{r}_0, t_0) \cdot \mathbf{V}(\mathbf{r}_0, t_0)$ as was done for the homogeneous case, using the well-known arguments based on the invariance of the phase.

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