The Group Doppler Effect

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ABSTRACT: *The Doppler effect for the modulation of a wave signal is investigated. This group Doppler effect is considered for rays in inhomogeneous, dispersive, absorbing, anisotropic and time-varying media. In addition, the receiver or scatterer is allowed to move with respect to the medium. It is shown that the new effects depend on the group velocity and its direction relative to the normal.*

I. Introduction

It is well known that the Doppler effect is a powerful tool for ionospheric research, see for example Kelso (**1**). There exists also numerous navigation and tracking methods based on this principle, see Gill (**2**). In these and other applications the main difficulty is the precise measurement of frequency and maintenance of a stable frequency reference. The frequency is usually prescribed by the propagation properties of the medium. On the other hand, high resolution frequency measurements by means of counters can be achieved in the kHz band mainly.

This suggests, in a natural manner, that the carrier frequency be modulated, and the Doppler effect measured by monitoring the wave envelope. The use of the group Doppler effect, as it is termed here, might therefore improve the performance of devices exploiting the Doppler effect. For example, in a bistatic scattering experiment, where the transmitter and receiver are not located at the same site, the reference group frequency can be transmitted over speech communication channels. As another example, one can think of a "two-media" Doppler experiment. The group frequency reaches an earth satellite by means of a carrier in the lower MHz band, where the ionosphere has a considerable effect on the wave propagation. The observed group frequency is detected and retransmitted to the ground station at microwave carrier frequency, where the effect of the ionosphere is negligible. This might be more tractable compared to a two-way Doppler experiment in the same medium.

It seems therefore worth while to investigate systematically the group Doppler effect and compare the results to the usual phase Doppler effect. In the following the group Doppler effect is investigated for rays propagating in inhomogeneous, dispersive, absorbing, anisotropic and time-varying media. If the receiver or scatterer is moving with respect to the medium, additional Doppler effects are encountered.

II. Basic Theory

It is assumed that the properties of the medium at hand vary slowly over distances and time intervals comparable to wavelength and period, respectively. This justifies the use of ray theory. However, there is another time scale involved, of the order of the "time of flight" of a wave packet from transmitter to receiver. By substitution of a plane wave solution into Maxwell's equations, we are led to the dispersion equation

$$F(\mathbf{k},\omega;\mathbf{r},t) = 0, \tag{1}$$

where $\mathbf{k}$ is the propagation vector and ω is the (angular) frequency. The uniqueness of the phase integral prescribes the Whitham conservation law

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla\omega = 0, \tag{2}$$

and the Sommerfeld–Runge law of refraction

$$\nabla \times \mathbf{k} = 0. \tag{3}$$

The group velocity is defined as

$$\mathbf{v} = \frac{\partial\omega}{\partial\mathbf{k}} = -\frac{\partial F/\partial\omega}{\partial F/\partial\mathbf{k}} = \mathbf{v}_g + i\mathbf{u}, \tag{4}$$

where $\partial/\partial\mathbf{k}$ denotes the gradient in $\mathbf{k}$ space and $\mathbf{u}$ is the imaginary part of $\mathbf{v}$, which is non-vanishing in absorbing media.

For moderate absorption Suchy (**3, 4**) shows that the velocity of a wave packet is given by $\mathbf{v}_g$, hence the path traversed by the wave packet is obtained by integrating $\mathbf{v}_g = \mathrm{d}\mathbf{r}/\mathrm{d}t$. In addition we have two pairs of equations [which are repeated here because of misprints in the original work (**4**)],

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_g}\operatorname{Re}\mathbf{k} - \mathbf{u}\cdot\frac{\partial}{\partial\mathbf{r}}\operatorname{Im}\mathbf{k} &= \operatorname{Re}\frac{\partial F/\partial\mathbf{r}}{\partial F/\partial\omega}, \\ \frac{\mathrm{d}}{\mathrm{d}t_g}\operatorname{Re}\omega - \mathbf{u}\cdot\frac{\partial}{\partial\mathbf{r}}\operatorname{Im}\omega &= -\operatorname{Re}\frac{\partial F/\partial t}{\partial F/\partial\omega}, \end{aligned}\right\} \tag{5}$$

and another two equations obtained by exchanging Im and Re and replacing $-\mathbf{u}$ by $\mathbf{u}$. Suchy (**4**) describes a method of integration, hence it is assumed that

$$\left.\begin{aligned} \omega &= \omega(\mathbf{r},t), \\ \mathbf{k} &= \mathbf{k}(\mathbf{r},t) \end{aligned}\right\} \tag{6}$$

are available along the known ray path.

Just as ω/k defines the phase velocity, relating the frequency and the wavelength, for a modulated wave we have

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\partial\omega}{\partial\mathbf{k}}\cdot\hat{\mathbf{k}} \approx \frac{\Delta\omega}{\Delta k}\,\hat{\mathbf{v}}\cdot\hat{\mathbf{k}}, \quad \hat{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1, \tag{7}$$

relating the frequency $\Delta\omega = \mathrm{d}\omega$ and wavelength $1/\Delta k$ of the envelope. The group Doppler effect is therefore defined as the change of $\mathrm{d}\omega$.

III. The Group Doppler Effect for Rays

Recently Censor and Brandstatter (**5**) discussed the phase Doppler effect in inhomogeneous time-varying media. These results are adequate for more general media. Due to temporal variations of the medium, wave packets emitted at different times will traverse different paths. Hence the ray path is defined as a two-parameter function

$$\mathbf{r} = \mathbf{R}(\tau, t), \tag{8}$$

where $t = \text{const}$ defines the ray path for a wave packet reaching the receiver at time t, which started to move at time $t-\tau$. Hence τ is the time of flight.

The phase Doppler effect is computed from the phase integral

$$\phi = \int_{(r_0,t_0)}^{(r,t)} (\mathbf{k}\cdot \mathrm{d}\mathbf{r} - \omega\, \mathrm{d}t) \tag{9}$$

integrated along a path in four-space. On any ray path (9) becomes

$$\left.\begin{aligned} \Phi(\tau,t) &= \int_\tau^0 \left[\mathbf{K}(\tau',t)\cdot\frac{\partial \mathbf{R}(\tau',t)}{\partial \tau'} + W(\tau' t)\right] \mathrm{d}\tau', \\ \mathbf{K}(\tau',t) &= \mathbf{k}(\mathbf{R}[\tau',t], t-\tau'), \\ W(\tau',t) &= \omega(\mathbf{R}[\tau',t], t-\tau'). \end{aligned}\right\} \tag{10}$$

This yields for the frequency measured by the observer [which is different from $\omega(\mathbf{r},t)$]

$$\omega^* = \omega(\mathbf{r}_0, t_0) + \int_0^\tau \frac{\partial}{\partial t}\left[\mathbf{K}(\tau',t)\cdot\frac{\partial \mathbf{R}}{\partial \tau'} + W(\tau',t)\right]\mathrm{d}\tau'. \tag{11}$$

We are concerned here with the group Doppler effect $\mathrm{d}\omega^*/\mathrm{d}\omega$, where $\omega = \omega(\mathbf{r}_0, t_0)$, hence

$$\frac{\mathrm{d}\omega^*}{\mathrm{d}\omega} = \frac{\mathrm{d}\omega^*/\mathrm{d}k}{\mathrm{d}\omega/\mathrm{d}k} = 1 + \frac{1}{\mathbf{v}_T\cdot\hat{\mathbf{k}}}\int_0^\tau \frac{\partial}{\partial t}\frac{\partial W}{\partial \mathbf{k}}\cdot\hat{\mathbf{k}}\,\mathrm{d}\tau', \tag{12}$$

where $\mathbf{v}_\tau = \mathbf{v}(\mathbf{r}_0, t_0)$. Since $(\partial/\partial t)\mathbf{K}$ is independent of $\mathbf{k}$, the first term of the integral in (11) vanished. From the dispersion equation (1) $\omega = \omega(\mathbf{k},\mathbf{r},t)$ is obtained, hence in (12) $W = W(\mathbf{k},\tau',t)$ and exchanging order of differentiation is allowed. But $(\partial W/\partial \mathbf{k})\cdot\hat{\mathbf{k}}$ is the projection of the group velocity on the wave normal direction at the point along the ray defined by τ'. Obviously for a time independent medium $\mathrm{d}\omega^*/\mathrm{d}\omega = 1$ and there is no group Doppler effect, which vanishes together with the phase Doppler effect. For absorbing media $\mathrm{d}\omega^*/\mathrm{d}\omega$ is complex, which may be interpreted as affecting changes in time of the amplitude of the envelope.

In a non-absorbing isotropic medium (12) can be obtained by an argument based on the time dilation along the ray path. In such a case, the path length for a wave packet emitted at $t-\tau$ and arriving at t is given by

$$p(\tau,t) = \int_0^\tau v(\mathbf{r}',t')\,\mathrm{d}\tau', \tag{13}$$

where $V(\tau', t) = v(\mathbf{R}[\tau', t], t-\tau')$, and the integration (13) can be carried out. Consider two wave packets, emitted at two points in space, at different time instances and moving along different paths, arriving at the receiver location at a time difference $\mathrm{d}t$. If the signals travelled along neighboring paths, then the difference in path length is

$$\mathrm{d}p = \frac{\partial p}{\partial \tau}\mathrm{d}\tau + \frac{\partial p}{\partial t}\mathrm{d}t. \tag{14}$$

If the signals have been emitted simultaneously, then $\mathrm{d}t = \mathrm{d}\tau$, thus from (14) we have $\mathrm{d}p/\mathrm{d}t$. The path difference is now interpreted as $\mathrm{d}p = v_\tau \overline{\mathrm{d}t}$, such that the two signals have been emitted on adjacent points on neighboring paths at a time lapse $\overline{\mathrm{d}t}$. As $\mathrm{d}t \to 0$ the two paths coalesce, defining the positions of transmitter and receiver and the ratio describing the time dilation is

$$\frac{\overline{\mathrm{d}t}}{\mathrm{d}t} = 1 + \frac{1}{v_\tau}\int_0^\tau \frac{\partial}{\partial t} V(\tau', t)\,\mathrm{d}\tau'. \tag{15}$$

If we argue that the group frequencies are inversely proportional to the time increments, then (15) becomes (12), for the special case of the isotropic case where the group velocity and wave normal are parallel.

In Ref. (**5**) an expression similar to (15) has been found for the time dilation in non-dispersive media. However, it was rejected as not representative of the phase Doppler effect, which was expected to emerge naturally from the phase function.

IV. Group Doppler Effect due to Moving Receiver and Scatterer

In addition to the group Doppler effect discussed above, which is due to time variations in the medium, we expect a group Doppler effect when the receiver or scatterer moves relative to the medium. Kelso (**1**), in a careful analysis based on the ray geometry, showed that for the phase Doppler effect the two phenomena are indeed additive. More generally, we consider the wave field in the vicinity of the moving receiver or scatterer, and expect the Doppler shift to be governed by the invariance of the phase.

The phase Doppler effect due to the motion of a receiver is given by

$$\omega' = \omega - \mathbf{k}\cdot\mathbf{U}, \tag{16}$$

where ω' is the frequency measured by the receiver, moving with a velocity $\mathbf{U}$ with respect to the medium. This result is relativistically exact to the first order in the velocity and suffices for all practical applications. Proceeding as before, we find

$$\frac{\mathrm{d}\omega'}{\mathrm{d}\omega} = \frac{\mathrm{d}\omega'/\mathrm{d}k}{\mathrm{d}\omega/\mathrm{d}k} = 1 - \frac{\mathbf{U}\cdot\hat{\mathbf{k}}}{\mathbf{v}\cdot\hat{\mathbf{k}}}. \tag{17}$$

Alternatively we could write

$$\frac{\mathrm{d}\omega'}{\mathrm{d}\omega} = \frac{\partial\omega'}{\partial\omega} + \frac{\partial\omega'}{\partial k}\frac{\partial k}{\partial\omega} \tag{18}$$

and recall that for anisotropic media

$$v \cos \alpha = \left(\frac{\partial k}{\partial \omega}\right)^{-1}, \tag{19}$$

where α is the angle subtended by $\mathbf{v}$ and $\mathbf{k}$. For isotropic media $\alpha = 0$ hence

$$\frac{d\omega'}{d\omega} = 1 - \mathbf{U}\cdot\hat{\mathbf{k}}/v \tag{20}$$

which can be obtained by a simplistic argument of counting wave crests, as presented for the phase Doppler effect in elementary textbooks. It is interesting to note that if $\mathbf{U}$ and $\mathbf{v}$ are parallel, such that the motion is along the ray, the direction of the wave normal in (17) is immaterial. For the special case $\mathbf{U}\cdot\hat{\mathbf{k}} = 0$ the group and phase Doppler effects vanish simultaneously.

In a scattering Doppler effect the receiver is replaced by a reflector which re-transmits the incident wave. The boundary conditions in the frame of reference co-moving with the object will prescribe that the frequencies of the incident and scattered waves be identical. Thus where the assumption of locally plane waves holds, the phase Doppler effect is prescribed by

$$\omega - \mathbf{k}\cdot\mathbf{U} = \omega^r - \mathbf{k}^r\cdot\mathbf{U}, \tag{21}$$

where superscript r denotes quantities pertaining to the reflected wave. Note that (21) is relativistically exact. By differentiation (21) yields

$$\frac{d\omega^r}{d\omega} = \frac{1 - \mathbf{U}\cdot\hat{\mathbf{k}}/\mathbf{v}\cdot\hat{\mathbf{k}}}{1 - \mathbf{U}\cdot\hat{\mathbf{k}}^r/\mathbf{v}^r\cdot\hat{\mathbf{k}}^r}. \tag{22}$$

For isotropic media $\mathbf{v}\cdot\hat{\mathbf{k}} = v$, $\mathbf{v}^r\cdot\mathbf{k}^r = v$ and (22) becomes simpler. This is the result one would expect by counting wave crests, as for the phase Doppler effect in simple cases. In the forward direction (along the ray) $\mathbf{v} = \mathbf{v}^r$ and the phase and group Doppler effect vanish simultaneously.

V. Conclusions

The group Doppler effect, i.e. the Doppler effect for the modulating signal, has been discussed. In the general case of a time-varying medium and moving observer, the total effect is given by

$$\frac{d\omega'}{d\omega} = \frac{d\omega'}{d\omega^*}\frac{d\omega^*}{d\omega} \approx 1 - \frac{\mathbf{U}\cdot\hat{\mathbf{k}}}{\mathbf{v}^*\cdot\hat{\mathbf{k}}^*} + \frac{1}{\mathbf{v}\cdot\hat{\mathbf{k}}}\int_0^{\tau}\frac{\partial}{\partial t}\mathbf{V}(\tau' t)\cdot\hat{\mathbf{k}}_{\tau'}\,d\tau', \tag{23}$$

which is a linearized form giving the first-order effects. Here the asterisk denotes quantities at the end of the ray, $\mathbf{v}, \hat{\mathbf{k}}$ pertain to the starting point and the quantities in the integral are taken at points along the ray. The group Doppler effect depends on the group velocity along the ray projected in the direction of the wave normal and on the component of the velocity $\mathbf{U}$ in this direction.

References

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(2) T. P. Gill, "The Doppler Effect", Logos Press and Academic Press, New York, 1965.
(3) K. Suchy, "The velocity of a wave packet in an anistropic absorbing medium", *J. Plasma Phys.*, Vol. 8, pp. 33–52, 1972.
(4) K. Suchy, "Ray tracing in an anisotropic absorbing medium", *J. Plasma Phys.* Vol. 8, pp. 53–65, 1972.
(5) D. Censor and J. J. Brandstatter, "Generalized Doppler effect in time varying media", *J. Franklin Inst.*, Vol. 297, pp. 485–490, 1974.

Note added in proof. Equations (5) as well as additional information is to be found in K. Suchy, "The propagation of wave packets in inhomogeneous anisotropic media with moderate absorption", *Proc. IEEE*, Vol. 62, pp. 1571–1577, 1974.