Exact Inverse-Separation Series for Multiple Scattering in Two Dimensions

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We consider configurations of arbitrary scatterers \((s = 1, \cdots, N)\) in two dimensions, such that the circles circumscribing the scatterers do not intersect. As shown previously [V. Twersky, in *Electromagnetic Waves*, R. E. Langer, Ed. (University of Wisconsin Press, Madison, 1962), pp. 361–389], the solution can be written in terms of the multiple-scattered scattering amplitudes \(G_s\) and the \(G_s\) are specified by the presumably known farfield isolated scattering amplitudes \(g_s\) by a set of integral equations \(G(g)\) (which can be converted to algebraic equations involving Hankel functions of the separations \(b_{sn}\), etc.). Among other applications, the previous paper gave the complete asymptotic series for \(G(g)\) in inverse powers of the \(b_s\); this was based essentially on Hankel’s asymptotic expansion for the Hankel functions \(H_n\). The present paper derives the analogous convergent representation of \(G(g)\) based on the exact representation of \(H_n\) in terms of Lommel polynomials. For \(N\) scatterers, we give the multiple-scattering solution as a series in \(H_0, H_1, b^{-1}\), and the derivatives of \(g\) with respect to angles. For two scatterers, we give a closed form in terms of a differential operator.

**INTRODUCTION**

Previous papers derived integral equations for multiple scattering of waves by configurations of arbitrary scatterers in two\(^1\) and in three\(^2\) dimensions. These equations specify the solution of the many-body problem in terms of the presumably known farfield scattering amplitudes (say \(g\)) of the scatterers in isolation. Equivalent representations as sets of algebraic equations were derived in terms of the corresponding isolated scattering coefficients (the coefficients of the Fourier, Mathieu, or Legendre series representations of \(g\)), and several applications were made. In particular, series forms of the solutions in inverse powers of the separations of scatterers were generated in terms of derivatives of \(g\) with respect to angles. These inverse-separation series, convergent in three dimensions and asymptotic in two dimensions, were based essentially on the corresponding inverse-distance (of observation) series for an isolated scatterer in two or three dimensions, or, equivalently, on Hankel’s series\(^3\) for the outgoing radial

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functions $H_\rho(r) = H^{(1)}_\rho(r)$ in inverse powers of $r$. If $\rho = n + \frac{1}{2}$, where $n$ is an integer, then Hankel's series reduces to $e^{i\rho}$ times a polynomial in $1/r$; however, if $\rho = n$, the representation is an asymptotic series; the corresponding scattering series are convergent in three dimensions and asymptotic in two dimensions. In the present paper, we derive the convergent inverse-separation series for multiple scattering in two dimensions.

For three-dimensional scattering problems, Sommerfeld\(^4\) considered the inverse-distance series form for a single scatterer and showed that the coefficients of successive terms could be obtained recursively from the first, i.e., from the usual scattering amplitude $g$. Wilcox\(^5\) showed that the series was uniformly and absolutely convergent outside the sphere circumscribing the arbitrary scatterer. Twersky\(^6\) recast the series as a differential operator on $g$ and used the resulting form to treat multiple scattering: for $N$ arbitrary scatterers (such that the spheres circumscribing the scatterers do not intersect), he obtained the multiple-scattering solution and multiple-scattering amplitude (say $G$) as a series in the $g$'s and the separations of the scatterers; for two, he obtained a closed form involving a differential operator for $G$ in terms of $g$.

For two-dimensional scattering problems, the analog of the Sommerfeld–Wilcox development has recently been given by Karp\(^7\), who used essentially the exact form of $H_\rho(r)$ in terms of $H_0(r)$ and $H_1(r)$ times Lommel polynomials in $1/r$ (see Ref. 3, p. 207). Karp\(^8\) showed that the representation in $H_0$ and $H_1$ and inverse powers of $r$ converged, and described a procedure for obtaining the coefficients of successive terms recursively from two values of the scattering amplitude. In the present paper, we construct the corresponding representation in terms of two differential operators, and then derive the convergent representation in inverse powers of separations for the multiple-scattering problem.

If the Hankel asymptotic expansions of $H_0$ and $H_1$ are inserted in the present results (the exact series for $N$ bodies and the closed operator form for 2 bodies), then we again obtain the complete asymptotic representations given previously.\(^1\) Although the asymptotic representations are the more useful ones even for only moderately large separations, the convergent ones bring the subject in two dimensions to the same level of completeness as in three, and are expected to facilitate obtaining explicit representations for specific problems.

### I. PRELIMINARY CONSIDERATIONS

We consider scattering problems such that the field outside the scatterer satisfies

$$\nabla^2 u + k^2 u = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad k = |k| = 2\pi/\lambda,$$

where $\psi$ consists of a plane-wave source term $\varphi$ and an outgoing scattered wave $u$:

$$\psi = e^{ikr} = e^{ikr \cos(\theta - \phi)} = ki, \quad r = ro;$$

$$u \sim g(\theta, \phi)e^{ikr - i\pi/(2\pi kr)} = g(\theta, \phi)H(kr), \quad r \to \infty.$$

In general, we take the origin of coordinates as the center of the smallest circle (radius $a$) circumscribing the scatterer. The scattering amplitude $g(\theta, \phi)$ (which is independent of $r$) specifies the "farfield" response in the direction $\theta$ to a wave incident in the direction $\phi$. The field satisfies any of the usual boundary or transition conditions at the scatterer's surface, but we need not consider these explicitly. Although we use scattering terminology, the results apply to any radiative solution of von Helmholtz's equations, i.e., to any solution of Eq. (1) subject to Eq. (3) (e.g., $g$ may be an antenna pattern function or a piston function).

Many different exact general representations for $u$ and $g$ exist: surface integrals, volume integrals, complex-integral spectral representations, infinite series of Bessel and circular functions, infinite series of Mathieu functions, as well as various "mixed representations" obtained by routine manipulations of basic forms. For $r > a$ (at the least), appropriate representations for $u$ and $g$ may be paired off as transforms, so that knowledge of either one determines the other.

In particular, if we apply Green's theorem to $u(\rho)$ and $H_0(k|\rho - \phi|)$ in the region external to the scatterer, we obtain the Helmholtz surface integral form

$$u(\rho, \phi) = \frac{1}{4i} \int [H_0(k|\rho - \phi|)\partial_\sigma u(\rho, \phi) - u(\rho, \phi)\partial_\sigma H_0]dS(\sigma),$$

where $S(\rho)$ is any surface inclosing the scatterer and excluding $r$, and $\partial_\sigma$ is the normal derivative outward.
from the scatterer (see Fig. 1). The corresponding representation for \( g \) of Eq. (3) follows from Eq. (4) by substituting the asymptotic form

\[
H_0(k|\mathbf{r}-\mathbf{q}|) \sim e^{-ikr\cos(\theta-\phi)}H(kr), \quad kr \gg 1, \ r \gg p. \tag{5}
\]

Thus,

\[
g(\theta, \alpha) = \left\{ e^{-ikr\cos(\theta-\phi)}u(\mathbf{r}, \mathbf{q}) \right\}, \tag{6}
\]

where the integral is over any surface inclosing the scatterer.

If \( u \) is known, then Eq. (6) gives \( g \) on integration. An inverse relation follows \(^1\) on introducing into Eq. (4) the Sommerfeld \(^4\) representation of \( H_0 \) as a complex integral of plane waves. As shown previously,\(^1\)

\[
u(\mathbf{r}, \alpha) = -\frac{1}{\pi} \int e^{ikr\cos(\theta-\phi)}g(\mathbf{r}, \mathbf{q})d\mathbf{r}, \tag{7}
\]

where the integral is over the Sommerfeld path \( \gamma+i\infty \) to \( \pi+i\infty \) with \( \gamma \) satisfying \( \cos(\theta-\gamma) - [\rho \sin(\phi-\gamma)]_{\text{max}} > 0; \) this path will be understood for all integrations over \( r \). Values of \( \rho \) on the scatterer's surface give the greatest range to \( \gamma \). If we take \( \gamma = \pi/2 \), then we require \( r > [\rho \cos(\theta-\phi)]_{\text{max}} \), i.e.,

\[
I = -\frac{1}{\pi} \int e^{i\tau \cos(\theta-\phi)}d\tau \sim H(r) \sum_{m=0} (-1)^m \frac{1}{(8\pi)^m} \\cdots (2m+1) \frac{1}{2^m} = \sum_{n=0} g_n \tag{10}
\]

in terms of the scattering amplitude \( g(\theta, \alpha) \) and its \( \partial \) derivatives. \(^1\) Terms to \( (kr)^{-4} \) were derived by Karp and Zitron \(^7\) essentially by substituting \( H_0(\mathbf{r})g + g/kr \) into the wave equation and solving for \( g \). We consider the exact series analogous to Eq. (10).

An alternative exact representation \(^1\) for \( u \) in terms of \( g \) may be constructed by using the addition theorem for \( H_0(k|\mathbf{r}-\mathbf{q}|) \) in Eq. (4), or by substituting the Fourier series

\[
g(\theta, \alpha) = \sum_{n=0} a_n(\alpha)e^{in\theta}, \tag{11}
\]

into (7) and isolating the usual integral representation

\[
\int_{-\infty}^{\infty} e^{i\tau \cos(\theta-\phi)}d\tau \sim H(r) \sum_{n=0} \frac{(1+4\pi^2)(9+4\pi^2) \cdots \Gamma(2m+1)^2 + 4\pi^2}{(8\pi)^m} \tag{12}
\]

of \( H \). This leads to

\[
u = \sum_{n=0} a_n(\alpha)i^n H_n(\mathbf{r})e^{in\theta}, \ r > a. \tag{13}
\]

The coefficients \( a_n \) (the Fourier, scattering, or multipole coefficients) determine both \( u \) and \( g \). We may eliminate \( a_n \) by substituting

\[
a_n(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} g(\mu, \alpha)e^{-i\pi n\mu}d\mu, \tag{13}
\]

to obtain

\[
u = \sum_{n=0} H_n(\mathbf{r})i^n e^{in\theta} \int_0^{2\pi} g(\mu, \alpha)e^{-i\pi n\mu}d\mu / 2\pi, \ r > a. \tag{14}
\]

The coefficients \( a_n \) may also be written as surface integrals by substituting Eq. (6) into (13) and replacing the plane wave by its Fourier-Bessel series; thus

\[
a_n(\alpha) = i^{-n}J_n(kr) e^{-i\pi n}u(\mathbf{r}, \alpha). \tag{14}
\]

The complete asymptotic series (10) for \( u \) may be obtained from Eq. (12) by using Eq. (8) and the differential equation satisfied by the exponential function:

\[
(\partial^2 + n^2)e^{in\theta} = 0. \tag{15}
\]
From Eqs. (8) and (15), we have
\[ i^s H_z(r)e^{i\theta} \sim D(r, \theta) = D(r, \theta) e^{i\theta}, \]
and consequently (14) reduces asymptotically to
\[ u \sim D(r, \theta) \sum e^{i\theta} \int_0^{2\pi} g(x, y) e^{-i\theta} d\theta = \frac{D(r, \theta)}{2\pi} g(\theta, \alpha) \]
as in (10). The analogous procedure for the three-dimensional problem,\(^2\) based on Hankel's polynomial representation for \( H_{n+1} \), yields \( u \) equal to \( e^{ikr}/ikr \) times an exact converging series in inverse powers of \( r \) (a series whose coefficients involve the scattering amplitude acted on by powers of Beltrami's differential operator).

In the range \( r > a \), Eqs. (7) and (14) are equivalent representations of the scattered field \( u \) (or of any radiative solution of the Helmholtz equation), and Eq. (17) is the corresponding complete asymptotic inverse-distance series. We obtained Eq. (7) from the basic form (4) and then obtained Eq. (14) from (7). We could just as well have obtained Eq. (14) from (4) (by using the addition theorem for \( H_0 \)) and then obtained Eq. (7) from (14) (by using the plane-wave representation of \( H_0 \)). (The sequence that we followed is ordered in terms of decreasing domain of validity in \( r \).) Similarly, the two procedures for obtaining the complete asymptotic representation are equivalent. We mentioned both and stressed the equivalence of the plane-wave and cylindrical-wave forms to facilitate a subsequent development. Additional discussion and representations are given in Ref. 1.

As discussed by Karp,\(^6\) a convergent inverse distance expansion for the two-dimensional field has essentially the form
\[ u(r, \theta) = P_0(kr) + \sum_{\nu=0}^{\infty} \frac{P_\nu(kr)}{\nu! (2\nu)!} \]
\[ r > a. \]

Karp showed that the series converge uniformly and absolutely for \( r > a \), and considered their analytical properties in detail. He pointed out that substituting Eq. (18) into the differential equation (1)—differentiating and rearranging the results as a linear combination of \( H_0 \) and \( H_1 \) and using the fact that the coefficients of the Hankel functions \( H_0 \) and \( H_1 \) must vanish—leads to \( P_\nu \) and \( Q_\nu \) recursively in terms of \( P_0 \) and \( Q_0 \); the zeroth-order coefficients were expressed in terms of \( g \) by comparison with asymptotic forms. As examples, Karp gave\(^6\)
\[ P_0(\theta) = \frac{[g(\theta) + g(\theta + \pi)]}{2}, \quad Q_0(\theta) = \frac{[g(\theta) - g(\theta + \pi)]}{2}. \]
\[ P_1 = -i \theta P_0, \quad Q_1 = \theta Q_0. \]
Thus, Karp\(^6\) has provided the parallel development for two dimensions of that followed by Sommerfeld\(^4\) and Wilcox\(^5\) for three dimensions. We now parallel Twersky's development\(^2\) for the three-dimensional problem and recast \( u \) explicitly in terms of \( \theta \) and \( g \); i.e., we obtain the convergent analog of Eq. (10).

II. EXACT OPERATIONAL FORM

As discussed in Watson,\(^3\) the Hankel function may be written exactly as
\[ H_n(r) = H_0(r) H_{n,0}(r) + H_1(r) H_{n,-1}(r), \]the general form of a Lommel polynomial is\(^3\)
\[ R_{n,\nu}(r) = \sum_{m=0}^{\infty} \frac{(-1)^m (n-m)!}{m! (n-2m)!} \frac{r^{2m-n}}{\Gamma(s+m)} \]
\[ \times \left[ (n+\nu)! (n+\nu-1)! \right] \]
\[ \frac{1}{(n-\nu)! (n-\nu-1)!}. \]
From this, we may represent the \( R \)'s of Eq. (20) as polynomials in \( n^2 \).

Corresponding to Eq. (20), and as determined by whether \( n \) is even or odd, we distinguish four cases of Eq. (21):
\[ R_{2n,0} = \sum_{\nu=0}^{n-1} C_{2n,\nu} \frac{(-1)^{n+\nu}}{(2\nu)! (2\nu)!} \]
\[ \times \left[ (n+\nu)! (n+\nu-1)! \right] \]
\[ \frac{1}{(n-\nu)! (n-\nu-1)!}. \]
\[ R_{2n-1,0} = \sum_{\nu=0}^{n-1} C_{2n-1,\nu} \frac{2 (n+\nu+1)}{(2\nu+1)} \]
\[ \frac{2 (2n+\nu+1)}{(2\nu+1)} \]
\[ \frac{2 \nu}{(n-\nu)! (n-\nu-1)!}. \]
In the quotient of factorials in brackets in \( R_{2n,0} \), we divide through by the denominator and pair off factors \( (n+m)(n-m) = n^2 - m^2 \) to write initially
\[ 2\nu \times \left[ (n+\nu) \cdots (n-\nu+1) \right] \times \left[ (2n+\nu) \cdots (2n-\nu) \right] \]
\[ = 2\nu \left[ (n^2 - \nu^2) \cdots [n^2 - (\nu - 1)^2] \right]; \]
doubling up once more and multiplying through by \( 2\nu \), we thus obtain
\[ [2n^2 \cdots (2n^2 - 2\nu^2)] \times \left[ (2n^2 - 2\nu^2) \cdots [2n^2 - (2n-2\nu)^2] \right] \]
\[ \times \left[ (2n^2 - 2\nu^2) \cdots [n^2 - (\nu - 1)^2] \right]; \]
with \( M_0 = 1 \). Equation (23) reduces the first two sets of \( R \)'s of Eq. (22) to polynomials in \( 2n^2 \). For \( R_{2n,1} \), we deal with the bracketed quotient of factorials times \( (n+\nu)/(n-\nu) \) and pair off factors \( 4(n+m)(n-m+1) \).
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\[(2n+1)^2-(2m-1)^2;\] doubling up, we obtain
\[
\{[(2n+1)^2-1][(2n+1)^2-9] \cdots [(2n+1)^2-(2r-1)^2]\}^2 \\
= N_\epsilon(-[2n+1]^3), \quad (24)
\]
with \(N_\epsilon=1\). Equation (24) reduces the last two sets of \(R\)'s of Eq. (22) to polynomials in \((2n+1)^2\).

In terms of Eqs. (23), (24), and
\[
I_\epsilon= (-1)^q/(2r)^{2q}(2q)!,
\]
we write the required coefficients as the polynomials

\[
R_\epsilon(-[2n]^3) = (-1)^q R_{2n,q} = \sum_{r=0}^\infty L_r M_\epsilon(-[2n]^3),
\]

\[
R_\epsilon'(-[2n]^3) = (-1)^q R_{2n-1,q} = -\sum_{r=0}^\infty L_r M_\epsilon'(-[2n]^3)
\]

\[
R_\epsilon'(-[2n+1]^3) = i(-1)^q R_{2n+1,0} = i \sum_{r=0}^\infty L_r N_\epsilon(-[2n+1]^3)
\]

\[
R_\epsilon(-[2n+1]^3) = i(-1)^q R_{2n,q} = i \sum_{r=0}^\infty L_r N_\epsilon'(-[2n+1]^3),
\]

where the upper limits were replaced by \(\infty\) since \([\text{from Eqs. (23) and (24)}]\) all new terms are identically zero. Thus, from Eqs. (20) and (26), we have

\[
(-1)^q H_{2n} e^{i(2n+1)\theta} = D_\epsilon(r; -[2n]^2),
\]

\[
i(-1)^q H_{2n+1} e^{i(2n+1)\theta} = D_{\epsilon'}(r; -[2n+1]^2).
\]

Note the factor of \(i\) incorporated in the odd values.

The essential feature of the above is that we have reduced \(H_{2n}\) and \(H_{2n+1}\) to polynomials in \((2n)^2\) and \((2n+1)^2\), respectively. Consequently, because of Eq. (15), we may replace the polynomials in \(n^2\) by ones in \(\delta^2 = \delta_2^2\) in the product forms; thus

\[
(-1)^q H_{2n} e^{i(2n+1)\theta} = D_\epsilon(r; -[2n]^2) e^{i(2n+1)\theta},
\]

\[
(-1)^q i H_{2n+1} e^{i(2n+1)\theta} = D_{\epsilon'}(r; -[2n+1]^2) e^{i(2n+1)\theta}.
\]

Thus, the Hankel–Fourier series (12) may be rewritten

\[
\sum a_{2n} e^{i(2n+1)\theta} = \frac{1}{2} \sum a_{2n} e^{i(n+1)\theta(1+e^{i\theta})} = \frac{1}{2} [g(\theta) + g(\pi + \theta)] = g_{\epsilon}(\theta),
\]

\[
\sum a_{2n+1} e^{i(2n+1+1)\theta} = \frac{1}{2} \sum a_{2n} e^{i(n+1)\theta(1-e^{i\theta})} = \frac{1}{2} [g(\theta) - g(\pi + \theta)] = g_{\epsilon'}(\theta),
\]

we have thus reduced Eq. (12) to the form

\[
u = a_\epsilon(r; \delta^2) g_{\epsilon}(\theta, \alpha) + a_{\epsilon'}(r; \delta^2) g_{\epsilon'}(\theta, \alpha).
\]

Here,

\[
D_\epsilon = H_\epsilon R_{\epsilon} + H_{\epsilon'} R_{\epsilon'},
\]

\[
R_{\epsilon'} = \frac{\delta^2 (\delta^2+4)^2}{2(2r)^3} - \frac{\delta^2 (\delta^2+4)^3 (\delta^2+16)^2}{4!(2r)^4} + \cdots,
\]

\[
R_{\epsilon'} = \frac{\delta^2 (\delta^2+4)^2}{2(2r)^3} - \frac{\delta^2 (\delta^2+4)^3 (\delta^2+16)^2}{5!(2r)^5} + \cdots,
\]
and

\[
D_o = H_0 R_o' + H_1 R_o,
\]

\[
R_o'/i = \frac{\partial^2 + 1}{2r} + \frac{\partial^2 + 9)(\partial^2 + 1)^2}{3!(2r)^3} - \frac{(\partial^2 + 25)(\partial^2 + 9)^2(\partial^2 + 1)^3}{5!(2r)^5} + \cdots,
\]

\[
R_o/i = 1 - \frac{\partial^2 + 1}{2(2r)^2} + \frac{(\partial^2 + 1)^2(\partial^2 + 9)^2}{41(2r)^4} + \cdots.
\]

The functions \(g_s\) and \(g_o\) are the components of the scattering amplitude \(g(\theta, \alpha)\) that are symmetrical and antisymmetrical with respect to reflection in the plane \(\theta + \pi/2\) (i.e., the plane perpendicular to the direction of observation).

If the scatterer is a monopole \(g(\theta, \alpha) = a_0\), then \(g_s = a_0\) and \(g_o = 0\); \(u\) reduces to \(D_o g_o\), and since all terms but the first of \(R_o\) involve \(\alpha^2\) we obtain simply \(u = H_0 a_0\) as required. If the scatterer is a monopole plus dipole \(g(\theta, \alpha) = a_0 + 2a_1 \cos \theta\), then \(g_s = a_0\) and \(g_o = 2a_1 \cos \theta\); the \(D_o g_s\) term again gives \(H_0 a_0\), and for \(D_o g_o\) [since all terms but the first involve \((\partial^2 + 1) \cos \theta = 0\)] we obtain \(H_1 i 2a_1 \cos \theta\). Thus \(u = H_0 a_0 + H_1 i 2a_1 \cos \theta\) as required.

Isolating \(H_o\) and \(H_1\) in Eq. (31), we have

\[
u = H_0 (R_o g_s + R_o g_o) + H_1 (R_o g_s + R_o g_o), \tag{34}
\]

which is the same form as Eq. (18) obtained by Karp.\(^6\) We now have the coefficients of Eq. (18) explicitly in terms of \(g\) and its \(\alpha\) derivatives:

\[
P_\alpha = (-1)^\mu M_s(\partial^2) g_s(\theta), \quad P_{\alpha+1} = -i(-1)^\mu [(\partial^2 + (2\alpha + 1)^2) N_s(\partial^2) g_s(\theta),
\]
\[
Q_\alpha = i g_o = [(\partial^2 + (2\alpha + 1)^2) M_s(\partial^2) g_o(\theta),
\]
\[
Q_{\alpha+1} = (-1)^\mu [(\partial^2 + (2\alpha + 1)^2) N_s(\partial^2) g_o(\theta),
\]

where \(M\) and \(N\) are the polynomials of Eqs. (23) and (24). Thus,

\[
P_0 = g_s - \frac{i(\partial^2 + 1)g_s(\theta)}{2kr} - \frac{\partial^3(\partial^2 + 4)g_s(\theta)}{2(2kr)^3} + \frac{(\partial^2 + 9)(\partial^2 + 1)^2g_s(\theta)}{3!(2kr)^3} + \cdots
\]

\[
Q_0 = i g_o = \frac{i(\partial^2 + 1)g_o(\theta)}{2kr} - \frac{\partial^3(\partial^2 + 4)g_o(\theta)}{2(2kr)^3} + \frac{(\partial^2 + 9)(\partial^2 + 1)^2g_o(\theta)}{3!(2kr)^3} + \cdots
\]

etc. The coefficients for 0 and 1 are Karp's results as in the present Eq. (19), and the other terms may also be obtained by the method that he describes.\(^6\) From Eqs. (18) and (36), we have

\[
u = H_0 \left\{ g_s - \frac{i(\partial^2 + 1)}{2kr} g_s - \frac{\partial^3(\partial^2 + 4)g_s}{2(2kr)^3} + \frac{(\partial^2 + 9)(\partial^2 + 1)^2g_s(\theta)}{3!(2kr)^3} + \cdots \right\}

\[
+ H_1 \left\{ i g_o - \frac{i(\partial^2 + 1)}{2kr} g_o - \frac{\partial^3(\partial^2 + 4)g_o(\theta)}{2(2kr)^3} + \frac{(\partial^2 + 9)(\partial^2 + 1)^2g_o(\theta)}{3!(2kr)^3} + \cdots \right\} \tag{37}
\]

For subsequent applications, we substitute the definitions of \(g_s\) and \(g_o\) of Eq. (30) and rewrite Eq. (31) as

\[
u = D(r; \partial^2) g(\theta, \alpha) + D'(r; \partial^2) g(\theta, \alpha), \quad D = (D_o + D_o)/2, \quad D' = (D_o - D_o)/2. \tag{38}
\]

If we replace \(H_o\) and \(H_1\) in Eq. (38) by their Hankel asymptotic series as in Eq. (8), then \(D \sim D_o\) as in Eq. (9) and \(D' \sim 0\). Thus Eq. (38) is a convenient form for assessing the corrections to the complete asymptotic representation (10).

Since the special function series (12) and the complex integral representation (7) are equivalent forms for \(r > a\), Eq. (31) is, of course, the exact series representation of Eq. (8) for \(r > a\), as well as of any other form of any solution of Eq. (1) subject to Eq. (3). Since Eq. (7) is general in form and since we may always write \(f(r) = [f(r) + f(r + \pi)]/2 + [f(r) - f(r + \pi)]/2 = f_s(r) + f_o(r)\), the above steps have shown that

\[
1/\pi \int e^{i r \cos(\theta - \alpha)} f(r) dr = \frac{1}{\pi} \int e^{i r \cos(\theta - \alpha)} [f_s(r) + f_o(r)] dr
\]

\[
= D_s(r; \partial^2) f_s(\theta) + D_o(r; \partial^2) f_o(\theta) = D(r; \partial^2) f(\theta) + D'(r; \partial^2) f(\pi + \theta), \tag{39}
\]

If we replace \(H_o\) and \(H_1\) in Eq. (38) by their Hankel asymptotic series as in Eq. (8), then \(D \sim D_o\) as in Eq. (9) and \(D' \sim 0\). Thus Eq. (38) is a convenient form for assessing the corrections to the complete asymptotic representation (10).
where \( \text{as in Eq. (7)} \) the limits are chosen to ensure convergence of the integral. Eq. (39) \([\text{as well as the previous Eq. (9)}]\) is a general result in terms of the parameter \( r \) and an analytic function \( f(r) \) such that \( f(\theta) \) is representable as a Fourier series. \([\text{The differential operator form of Eq. (39) may also be obtained by writing} f(r) \text{as a series in} e^{inr}, \text{expressing} e^{inr} \text{as polynomials in} \sin r, \text{using}\ e^{ir\cos\alpha} \sin r = (i/r)\partial_r e^{ir\cos\alpha}, \text{etc., and integrating by parts.}] \) In Sec. III, we apply Eq. (39) to the special case where \( f \) is a product of scattering amplitudes.

### III. MANY SCATTERERS

For a plane wave \( \psi = e^{ikr\cos(\theta - \alpha)} \) incident on many \((t = 1, 2, \ldots, N)\) arbitrary scatterers located at \( b_t \) in the geometry of Fig. 2, it was shown previously\(^1\) that the scattering solution of Eq. (1) could be written as

\[
U_t = \varphi_t U_t(\theta - \phi_t),
\]

\[
U_t(\theta) = \sum U_t(\theta - \phi_t),
\]

where \( \theta \) is the vector from the “center” of scatterer \( t \), and where \( U_t \) is the multiple-scattered wave of scatterer \( t \), and \( G_t \) is its corresponding scattering amplitude. As the neighbors of scatterer \( t \) recede to infinity, \( U_t \) and \( G_t \) reduce to \( \varphi_t U_t \) and \( \varphi_t G_t \), where \( \varphi_t = e^{ikr_t \cos(\theta - \alpha_t)} \) and where \( \varphi_t \) and \( G_t \) are the appropriate functions for scatterer \( t \) in isolation. If \( kr_0 \to \infty \), then \( U_t \to \sum H(kr_t)G_t(\theta, \alpha) = H(kr)G_t(\theta, \alpha) \times e^{-ikr \cos(\theta - \alpha)} \); we may also work\(^1\) with \( G_t = G_t(\theta, \alpha) \) in order to make the phase of \( \varphi \) more explicit. The provinces of the different representations for \( U_t \) are essentially as discussed for \( \varphi \) in Sec. I.

If we use the plane-wave form of Eq. (40) for the excitation at \( t \) arising from all neighbors \( \text{(i.e., for} \ \sum U_t, \text{where the prime means} s \neq t \) and the superposition principle for the total field scattered by \( t \) in response to a set of plane waves, we obtain\(^1\) the basic set of self-consistent integral equations

\[
G_t(\theta, \alpha) = g_t(\theta, \alpha) \varphi_t + \sum_{s \neq t} e^{ikb_{ts} \cos(\theta_s - \alpha)} g_t(\theta, \alpha) G_s(\tau', \alpha) d\tau'/\pi,
\]

where \( b_{ts}(b_{ts} + b_{ts}) = b_t - b_s \). The limits on the integrals, essentially as in Eq. (7), require\(^1\) \( b_{ts} \sin(\gamma - \beta_{is}) + \rho_s \sin(\gamma - \varphi_s) - \rho_s \sin(\gamma - \varphi_s) < 0 \); if \( \rho_t \) and \( \rho_s \) are on the appropriate scatterers’ surfaces and if we take \( \gamma = \beta_{is} - \pi/2 \), then we require\(^1\) that the separation of scatterer “centers” \( (b_{ts}) \) be greater than the sum of the scatterer’s projections on \( b_{ts} \). For \( b_{ts} > a_t + a_s \) essentially as for Eq. (14), we may substitute a series \( \sum B_{ts} e^{in\theta} \) for \( g_t G_s \) and isolate the integral representation of \( H_s \) to obtain

\[
G_t(\theta, \alpha) = g_t(\theta, \alpha) \varphi_t + \sum_{s \neq t} \sum_B H_n(kb) e^{in\theta} d\mu/2\pi.
\]

Equation (42) could also have been constructed by using the addition theorem for \( H_n(kr_t + b_{ts}) \) in (40) and taking the plane-wave form of the resulting \( J \)’s as the excitation—i.e.,

\[
U_s(\tau) = U_s(\tau + b) = \sum_{n} H_n(kb) i^{n} e^{i\theta} \sum_{s} e^{-i\tau} J_{s-n}(k) e^{i(\tau - \pi)n} d\tau'/\pi,
\]

where \( b_{ts}(b_{ts} + b_{ts}) = b_t - b_s \). The limits on the integrals, essentially as in Eq. (7), require\(^1\) \( b_{ts} \sin(\gamma - \beta_{is}) + \rho_s \sin(\gamma - \varphi_s) - \rho_s \sin(\gamma - \varphi_s) < 0 \); if \( \rho_t \) and \( \rho_s \) are on the appropriate scatterers’ surfaces and if we take \( \gamma = \beta_{is} - \pi/2 \), then we require\(^1\) that the separation of scatterer “centers” \( (b_{ts}) \) be greater than the sum of the scatterer’s projections on \( b_{ts} \). For \( b_{ts} > a_t + a_s \) essentially as for Eq. (14), we may substitute a series \( \sum B_{ts} e^{in\theta} \) for \( g_t G_s \) and isolate the integral representation of \( H_s \) to obtain

\[
G_t(\theta, \alpha) = g_t(\theta, \alpha) \varphi_t + \sum_{s \neq t} \sum_B H_n(kb) e^{in\theta} d\mu/2\pi.
\]
however, instead of working with such mixed forms (cylindrical waves to a given point and plane waves in the neighborhood of the point), it is, in general, more convenient to work with a pure plane-wave form, such as

\[ U_i(r, b) = \int e^{i k b \cos(\theta-\phi) + i k r \cos(\theta-\phi)} G_i(\tau, \alpha) d\tau / \pi, \tag{44} \]

which was used for Eq. (41), and then transform the result. See Ref. 1 for additional discussion, for various sets of algebraic equations derived directly from Eq. (41) by substituting Fourier-series forms, or Mathieu-series for the amplitudes, and for various applications.

In particular, the complete asymptotic representation for spacings \(kb \gg 1\) is obtained on applying Eq. (9) to the integrals in (41), or equivalently on applying Eq. (16) to the sum over \(n\) in (42) and proceeding as for (17):

\[ G_i(\theta, \alpha) \sim g_i(\theta, \alpha) + \sum_{\nu} D_{\nu} g_i(\theta, \beta_{\nu}, \alpha) G_i(\beta_{\nu}, \alpha), \tag{45} \]

where \(D_{\nu} = \Delta(k b \nu; \beta_{\nu}^2) / \beta_{\nu}^2\) is defined in Eq. (9). The form (45) is most convenient for generating expansions for even moderately large spacings, and has also been applied to obtain a closed operational form in terms of \(D\) for two scatterers. It also holds for combinations of scatterers and sources.

The analog of Eq. (45) in terms of the exact representation Eq. (39) is

\[ G_i(\theta, \alpha) = g_i(\theta, \alpha) + \sum_{\nu} D_{\nu} g_i(\theta, \beta_{\nu}, \alpha) G_i(\beta_{\nu}, \alpha), \tag{46} \]

where \(D_{\nu} = D(k b \nu; \beta_{\nu}^2) / \beta_{\nu}^2\) stands for either of the two differential operator forms of Eq. (39). Thus, we may write the summand of Eq. (46) essentially as

\[ D_{\nu} g_i(\theta, \beta_{\nu}, \alpha) G_i(\beta_{\nu}, \alpha) = D_{\nu} g(\theta, \pi + \beta) G(\pi + \beta, \alpha), \tag{47} \]

or, equivalently, as

\[ D_{\nu} [g(\theta, \beta) G(\beta, \alpha) - g(\theta + \beta, \pi + \beta)] = D_{\nu} [g(\theta, \beta) G(\beta, \alpha) - g(\theta, \pi + \beta)] \tag{48} \]

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In general, it is more convenient to work with representation (47). The matrix form

\[ \begin{bmatrix} D_{\nu} g_i(\theta, \beta_{\nu}, \alpha) G_i(\beta_{\nu}, \alpha) \\ D_{\nu} g(\theta, \pi + \beta) G(\pi + \beta, \alpha) \end{bmatrix} \tag{49} \]

is particularly convenient. The two elements of the column vector \([G_i(\theta, \alpha)]\) are required to represent \(U_i\) in the form Eq. (38), and Eq. (49) can be iterated directly.

Iterating Eq. (49), we obtain

\[ [G_i(\theta, \alpha)] = [g_i(\theta, \alpha) \phi_i] + \sum_{\nu} [D_{\nu} g_i(\theta, \beta_{\nu}, \alpha)] [g_i(\beta_{\nu}, \alpha)] \]

\[ + \sum_{\nu} [D_{\nu} g_i(\theta, \beta_{\nu}, \alpha)] \sum_{\mu} [D_{\nu} g_i(\beta_{\nu}, \beta_{\mu}, \alpha)] \]

\[ \times [g_i(\beta_{\mu}, \alpha) \phi_i] + \cdots, \tag{50} \]

where \([Dg]\) is the square matrix of Eq. (49), and the other terms are column vectors. The convergence properties of the iterated series of Eq. (50) (the "orders of scattering" series) must be based primarily on \([g]\). Only if \(kb \gg 1\) (as for the asymptotic form) would we regroup as an expansion in inverse powers of \(b\); using the asymptotic forms of \(H_0\) and \(H_1\), we have \(D \sim D' \sim 0\), and the representations for \(G_i\) of Eqs. (49) and (50) go over directly to (45) and to its iterations discussed previously.

IV. TWO SCATTERERS

For two scatterers, we can eliminate \(G_i\) from the right-hand side of Eq. (46) and obtain \(G(\theta)\) in closed operator form. We take the primary reference origin \((r=0)\) as the midpoint of the line joining the centers of the circles circumscribing the scatterers and locate the centers by

\[ b_+(\beta_+ \alpha) = b_+ (\beta, \alpha) = b_+(\beta, \pi + \beta), \]

\[ b_-(\beta_- \alpha) = b_-(\beta, \alpha) = b_-(\beta, \pi + \beta). \]

In terms of the local coordinates \(r_+\) and \(r_-(\text{the vectors from the centers})\), we have \(u = U_r(\tau_+) + U_r(\tau_-). \) For \(k r_+ \gg 1\) and \(r_\pm \gg b\), we have

\[ u = \Delta(k r) g(\theta, \alpha), \tag{51} \]

\[ G(\theta, \alpha) = g(\theta, \alpha) + \sum_{\nu} D_{\nu} g(\theta, \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) = g(\theta, \alpha) + \sum_{\nu} D_{\nu} g(\theta, \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) \tag{52} \]

whose asymptotic form was discussed previously in detail. As in Eq. (49), \([Dg]_{\alpha} = \Delta(k r) g(\theta, \alpha) + \sum_{\nu} D_{\nu} g(\theta, \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) \]

\[ = \Delta(k r) g(\theta, \alpha) + D_{\nu} g(\theta, \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) \tag{53} \]

Substituting for \(d\) in Eq. (52), we obtain initially

\[ [G_+(\beta_+ \alpha)] = [g_+(\beta_+ \alpha) \phi_+] + [D_{\nu} g_+(\beta_+ \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) + D_{\nu} g_+(\beta_+ \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) \tag{54} \]

Iterating once yields

\[ [G_+(\beta_+ \alpha)] = [g_+(\beta_+ \alpha) \phi_+] + [D_{\nu} g_+(\beta_+ \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) + D_{\nu} g_+(\beta_+ \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) \tag{55} \]

\[ + [D_{\nu} g_+(\beta_+ \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) + D_{\nu} g_+(\beta_+ \beta_{\nu}, \alpha) G(\beta_{\nu}, \alpha) \tag{56} \]
Thus, eliminating \([G_T(\beta_\pm,\alpha)]\) from the right-hand side, we obtain

\[
\begin{align*}
\frac{[G_T(\beta_\pm,\alpha)]}{[I - [D_T g_T(\beta_\pm,\beta_T)][D_{AB}(\beta_\pm,\beta_T)]^{-1}]} & \times \frac{[g_T(\beta_\pm,\alpha)\varphi_\pm]}{[D_T g_T(\beta_\pm,\beta_T)][g_T(\beta_\pm,\alpha)\varphi_\pm]}.
\end{align*}
\]

(55)

The inverse matrix \([I - AB]^{-1}\) expands as \(I + AB + ABAB\), we work from right to left in performing the \(D\) operations in the generated “chains,” and the phase factors \(\varphi_\pm\) are not operated on.

Substituting Eq. (55) into (52) yields the sought for closed operational form \([G(g)]\), which, together with Eq. (38), provides the complete representation for the multiple-scattered functions \(U_\pm\) in terms of \(H_0, H_1\) and the isolated scattering amplitudes \(g_\pm\). If \(D \sim B\), then the present result reduces to the previous asymptotic one. If we specialize the present \(g\)'s to those corresponding to different pairs of circular cylindrical monopoles, dipoles, or monopoles plus dipoles, then the present result goes over to the special closed forms in terms of \(H_0, H_1\) [with \(H_0 + H_1\) replaced by \(H_1(2kb)/kb\)] given previously. For the corresponding pairs of multipoles of the elliptic cylinder, the previous closed forms in terms of the radial Mathieu functions are now developed as series in \(H_0\) and \(H_1\).