Received 26 June 1970 11.2, 11.5, 11.7

Scattering of Elastic Waves by Moving Objects

JACOB ABOUDI* AND DAN CENSOR

Department of Environmental Sciences, Tel-Aviv University, Ramat-Aviv, Israel

It is shown that simple elastic media in motion exhibit new properties, owing to the effective compressional and shear wave velocities produced by the motion. Presently, we consider scattering of a compressional plane wave by (1) a half-space moving parallel to the interface and (2) a cylinder moving along its axis. In both cases the assumed boundary conditions correspond to good contact at the interface. Computational results are given for the scattering amplitudes as a function of velocity and angle of incidence.

INTRODUCTION

Scattering and propagation in acoustic moving media are discussed by Morse and Ingard.[1] Reflection and transmission at a plane interface are considered by Keller,[2] Miles,[3] and Ribner.[4] Yeh[5,6] discusses the moving fluid layer problem. Graham and Graham[7] consider the effect of the transition boundary layer between the two regions. Scattering of acoustic waves by a cylinder moving along its axis is given by Yeh.[8] Some references concerning the analogous electromagnetic problem are given by Censor.[9,10] Analogous problems for moving elastic scatterers are considered here. In contradistinction to the acoustic problem, two effective parameters are encountered here, namely, the effective compressional and shear wave velocities.

The correct choice of boundary conditions at the moving interface is a severe problem. In a concrete case, one expects a lubricating (viscous) layer between the interfaces, which involves nonuniform velocity within this region. Thus Graham and Graham[7] consider an acoustical problem with linear velocity profile, but without applying the boundary conditions relevant to viscoelasticity. Even without the correct viscoelastic boundary conditions, the problem becomes too complicated for close-form representation.[7] Since this problem is not the main theme here, we compromise by considering limiting cases. The case of very good lubrication implies that normal stresses and displacements at the interface are continuous, but that tangential stresses vanish. This has been considered elsewhere[11] for the configuration of a moving elastic slab. Here we assume the opposite situation, i.e., a very poor lubrication, which in the limiting case implies that stresses and displacements, both normal and tangential, are transmitted across the boundary. This is a fairly general model, since by letting the shear-wave velocity in some region vanish, elastic–acoustical problems are obtained as special cases.

Besides the theoretical interest, some engineering applications might be relevant: for example, probing a moving fluid inside an elastic duct, which is not otherwise accessible, or scattering by elastic moving objects within fluids.

I. STATEMENT OF PROBLEM

We consider the problem of scattering of plane, space-, and time-harmonic elastic compressional (P) waves by objects moving parallel to the boundaries: (a) a moving half-space and (b) a cylinder moving along its axis. The objects are moving in the direction $\hat{\mathbf{z}}=\mathbf{z}/z$ (see Fig. 1) with a constant velocity v, as observed in frame of reference xyz, at rest with respect to the external medium.

The incident compressional wave has the form

$$\phi_0=\exp(i\mathbf{k}_p\cdot\mathbf{r}-i\omega t),\quad \mathbf{k}_p\cdot\mathbf{r}=k_p(z\cos e+x\sin e). \quad (1)$$

Subsequently, the time dependence $\exp(-i\omega t)$ will be suppressed throughout. In Eq. 1, ϕ_0 is the displacement potential of the incident wave, $k_p=\omega/\alpha_1$ is the propagation constant, α_1 is the compressional wave velocity in the external medium denoted by 1, and ω is the angular frequency (henceforth, the frequency).

The boundary conditions at the interface are the continuities of the displacements and stresses. To satisfy these conditions, similarly as in the case of media at rest, reflected and transmitted compressional and shear (S) waves are required. Accordingly, the scattered fields can be generally represented as a

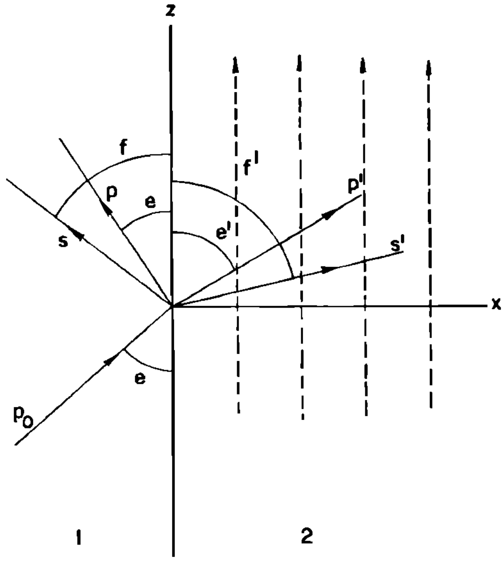

FIG. 1. Scattering of an incident compressional wave by a moving elastic half-space. The dashed arrows show the direction of motion of medium 2, as seen by an observer at rest in medium 1.

superposition (integral) of plane waves:

$$\begin{aligned}
\phi_1 &= \int \exp(i\mathbf{k}_p\cdot\mathbf{r})g_p(\tau)d\tau, \quad \hat{\mathbf{k}}_p\cdot\hat{\mathbf{z}}=\cos e, \\
&\qquad \hat{\mathbf{k}}_p\cdot\hat{\mathbf{x}}=\cos\tau, \\
\psi_1^{(i)} &= \int \exp(i\mathbf{k}_s\cdot\mathbf{r})g_s^{(i)}(\tau)d\tau, \quad \hat{\mathbf{k}}_s\cdot\hat{\mathbf{z}}=\cos f, \\
&\qquad \hat{\mathbf{k}}_s\cdot\hat{\mathbf{x}}=\cos\tau, \\
\phi_2 &= \int \exp(i\mathbf{K}_p\cdot\mathbf{r})G_p(\tau)d\tau, \quad \hat{\mathbf{K}}_p\cdot\hat{\mathbf{z}}=\cos e', \\
&\qquad \hat{\mathbf{K}}_p\cdot\mathbf{x}=\cos\tau, \\
\psi_2^{(i)} &= \int \exp(i\mathbf{K}_s\cdot\mathbf{r})G_s^{(i)}(\tau)d\tau, \quad \hat{\mathbf{K}}_s\cdot\hat{\mathbf{z}}=\cos f', \\
&\qquad \hat{\mathbf{K}}_s\cdot\hat{\mathbf{x}}=\cos\tau, \\
&\qquad i=1,2,
\end{aligned} \tag{2}$$

where $k_s=\omega/\beta_1$ and β_1 is the shear wave velocity in medium 1; similarly, K_p and K_s are effective propagation constants depending on the velocity, as described later. We denote by ϕ and ψ the displacement potentials for the compressional and shear waves, respectively. The functions g_p and $g_s^{(i)}$ are the scattering amplitudes for the potential at hand; in Eq. 2 they indicate the amplitude of the plane wave propagating in the variable direction τ. In the case of two half-spaces, the integrals reduce to single plane waves, and only $i=1$ is relevant.

The velocity of medium 2 affects the apparent wave velocities according to

$$\begin{aligned}
\alpha_{\text{eff}} &= \alpha_2 + v\cos e' = \omega/K_p, \\
\beta_{\text{eff}} &= \beta_2 + v\cos f' = \omega/K_s,
\end{aligned} \tag{3}$$

where α_2 and β_2 are the compressional and shear-wave velocities, respectively, in medium 2 at rest. Equations 3 follow from the transformation of the Helmholtz wave equations for the potentials ϕ_2 and ψ_2 into the frame of reference of medium 1 at rest, exploiting the fact that the boundary conditions preserve the frequency of the incident wave in this frame of reference. The first of Eqs. 3 appears in the previously cited references dealing with the corresponding acoustical problem, which follows as a special case upon putting $\beta_1\to 0$, $\beta_2\to 0$.

II. MOVING HALF-SPACE

The geometry for scattering by a moving half-space is shown in Fig. 1. Expressions for the displacement and stress components in terms of the potentials ϕ and ψ are given, for example, by Ewing, Jardetzky, and Press.[12] Using these expressions and applying the boundary conditions leads to Snell's law:

$$\frac{\cos e}{\alpha_1}=\frac{\cos f}{\beta_1}=\frac{\cos e'}{\alpha_{\text{eff}}}=\frac{\cos f'}{\beta_{\text{eff}}}\equiv\frac{\xi}{\omega}, \tag{4}$$

and to four equations relating the reflection and transmission coefficients (see pp. 83 ff. of Ref. 12). Let us denote by A_1 and B_1 the reflection coefficients for the P and S waves, respectively, and similarly A_2 and B_2 for the waves transmitted in medium 2. The explicit expressions for the reflection and transmission coefficients can be derived from pp. 83 ff. of Ref. 12, taking α_{eff} and β_{eff} according to Eqs. 3 for the moving region.

To investigate Snell's law (Eq. 4) without loss of generality it suffices to consider acute angles of incidence $0<e<\pi/2$ and $-\infty<v<\infty$, since this covers all possible cases. A critical angle $e=e_c$ will occur for P or S waves when e' or f', respectively, vanish; e.g., for P waves,

$$\cos e_c=\alpha_1/(\alpha_2+v). \tag{5}$$

If, for example, the velocity is normalized, and the same materials are used in regions 1 and 2 ($\alpha_1=\alpha_2=1$), then it follows from Eq. 5 that criticality is possible only for $v>0$. For $e<e_c$, we are within the shadow zone.

In the usual case of $v=0$, we have $e'<f'$. This follows from the fact that $\beta<\alpha$. From Eq. 4 it is established that this statement is valid for arbitrary velocities too.

For identical media and $v=0$ we get $e=e'$; this expected result follows from Eqs. 3 and 4. For $v>0$, we have $e'<e$, until criticality occurs. For $v<0$, we get $e'>e$ until, for $v\to-\infty$, the angle e' tends to $\pi/2$, for arbitrary angles e. In every case $f'>e'$, as mentioned above. For a given angle of incidence e, let us investigate the behavior of the transmitted waves

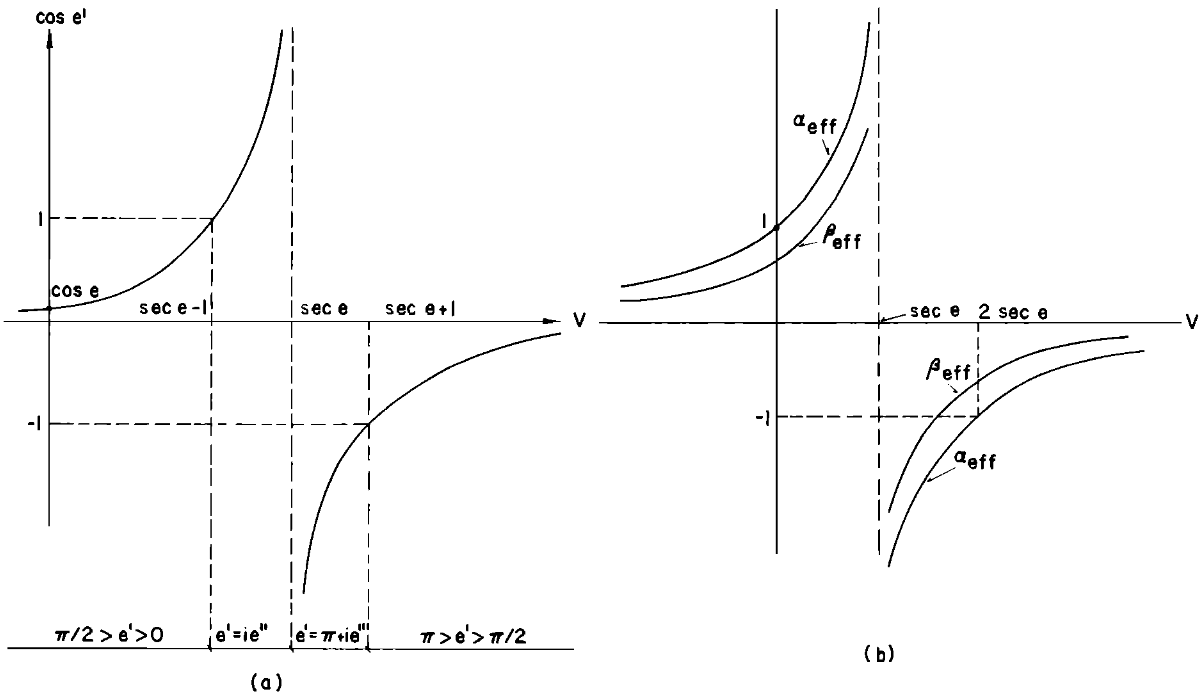

FIG. 2. Sketch of (a) $\cos e'$ and (b) the effective velocities, as functions of v. The chosen compressional wave velocities are $\alpha_1=\alpha_2=1$, and the shear wave velocities are $\beta_1=\beta_2=(3)^{-\frac{1}{2}}$.

as a function of the velocity v according to

$$\cos e'=\cos e/(1-v\cos e), \tag{6}$$

which follows from Eqs. 3 and 4 with $\alpha_1=\alpha_2=1$.

The above discussion describes the behavior for the range $-\infty<v\leqslant v_0$, where $v_0=(1/\cos e)-1$, i.e., the velocity for which criticality occurs for a given angle of incidence e (see Fig. 2). For $v_0<v<v_0+1$, we have $\cos e'>1$, which implies $e'=ie''$, where e'' is real. In addition, e'' must be chosen positive in order to get an exponentially decaying wave as a function of x (Fig. 1). In this range the behavior of the reflection and transmission coefficients is the same as for the case of the different media at rest when criticality takes place. For $v_0+1<v<v_0+2$, we have $-\infty<\cos e'<-1$. Therefore, $e'=\pi+ie'''$, where e''' is real and positive. This ensures again an attenuated wave. Note that in this range the effective velocity given by Eqs. 3 is negative. For $v_0+2<v<\infty$, we have $-1<\cos e'<0$, i.e., $\pi/2<e'<\pi$. In this range there is no criticality; i.e., the transmitted wave has a real direction of propagation. This phenomenon takes place at supersonic velocities $v>\alpha_2$. The same argument applies to $\cos f'$ too. It is noted that α_{eff} and β_{eff} become infinite at the same velocity v_0+1 [see Fig. 2(b)].

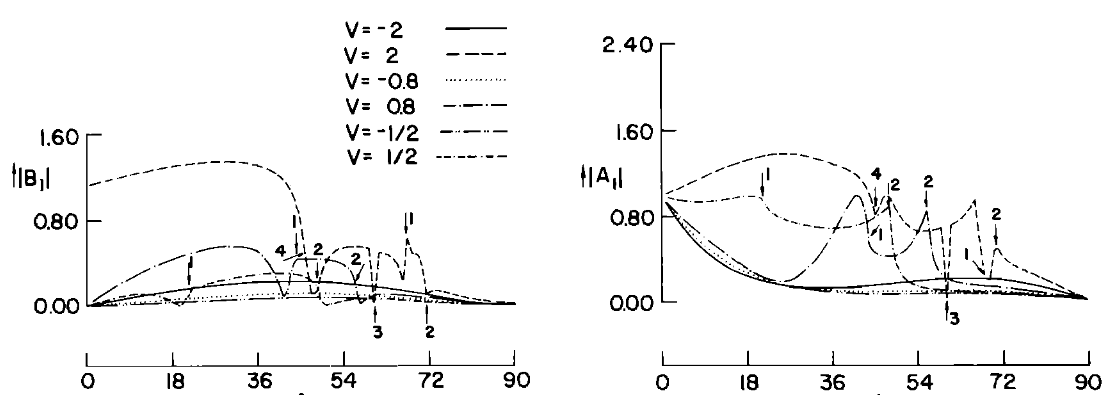

FIG. 3. Absolute values of the reflection coefficients for (a) compressional and (b) shear waves, due to an incident compressional wave. The wave velocities are as for Fig. 2.

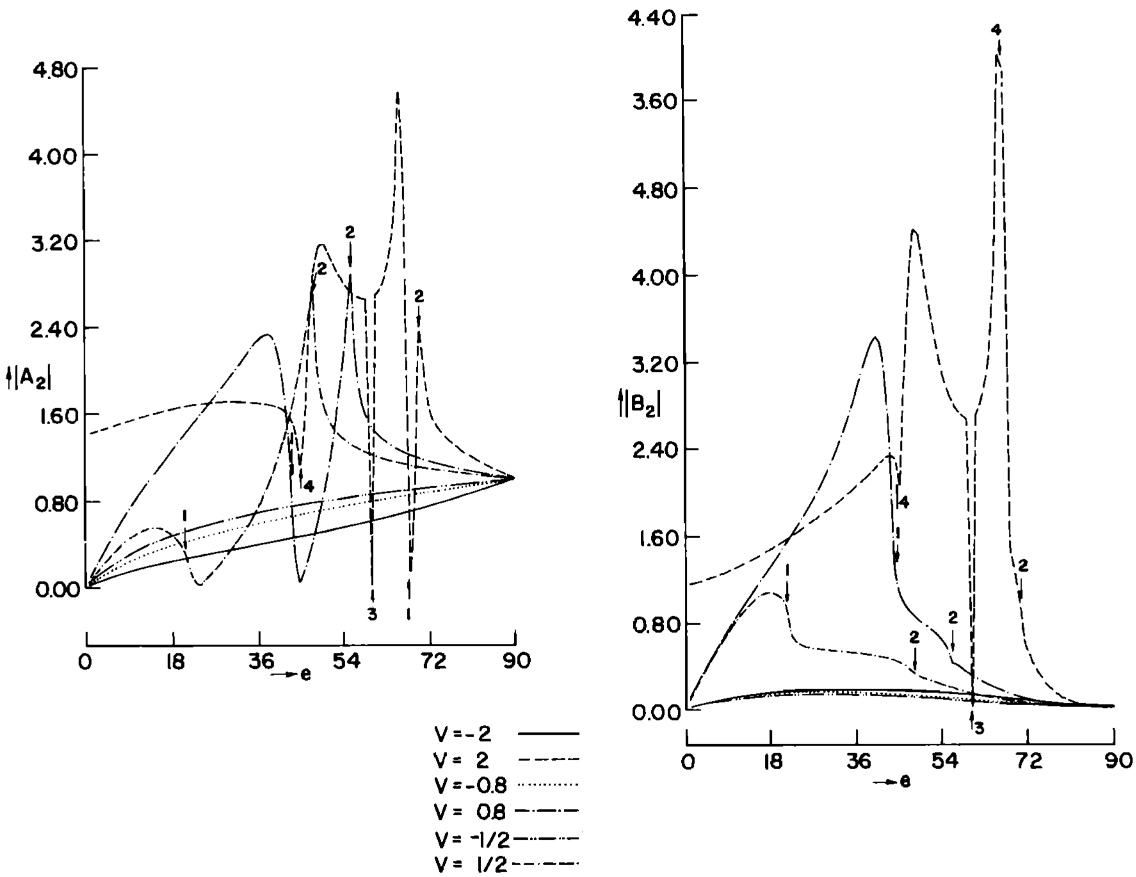

FIG. 4. Absolute values of the transmission coefficients for (a) compressional and (b) shear waves, due to an incident compressional wave. The wave velocities are as for Fig. 2.

In Figs. 3 and 4, reflection and transmission coefficients, respectively, are given for various velocities as a function of the angle of incidence $0<e<\pi/2$. It is assumed that the two media are identical when at rest. In the present case, $\alpha_1=\alpha_2=1$ and $\beta_1=\beta_2=\frac{1}{3}\sqrt{3}$ have been chosen; thus the Lamé constants are identical.

For negative velocities no criticality occurs, as mentioned above, and the curves show a smooth behavior as a function of e. On the other hand, for positive velocities, marked effects are apparent for angles of incidence where criticality occurs (i.e., $|\cos e'|=1$), and when the effective velocities attain an infinite value. In these curves, point 1 indicates the angle e for which criticality occurs for the transmitted S wave, such that to the left of this point we are within the range of criticality. Similarly, point 2 indicates that for higher e there is no criticality for the transmitted P waves. At point 3, $\alpha_{\text{eff}}=\beta_{\text{eff}}=\infty$, and $\cos e'$ and $\cos f'$ change sign from $-\infty$ to $+\infty$ as e increases. Point 4 indicates that, for higher e, criticality occurs for refracted S waves. This happens for $\cos f'=-1$. The range where both $-1<\cos f'<0$ and $-1<\cos e'<0$, which would be at higher velocities, is not depicted in

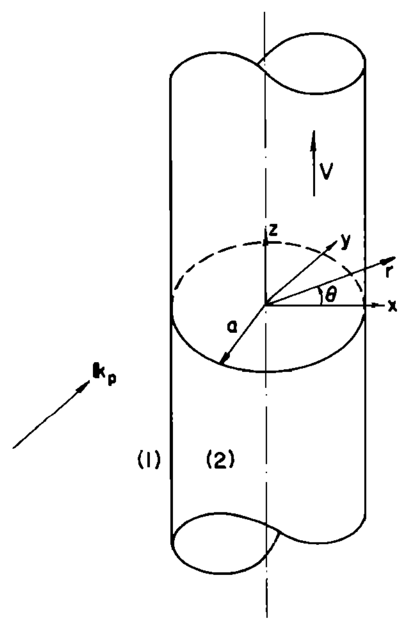

FIG. 5. Geometry for scattering of an incident compressional wave by a moving cylinder. The external medium 1 is at rest. Observed from this frame of reference, the cylinder is moving in the z direction with velocity v.

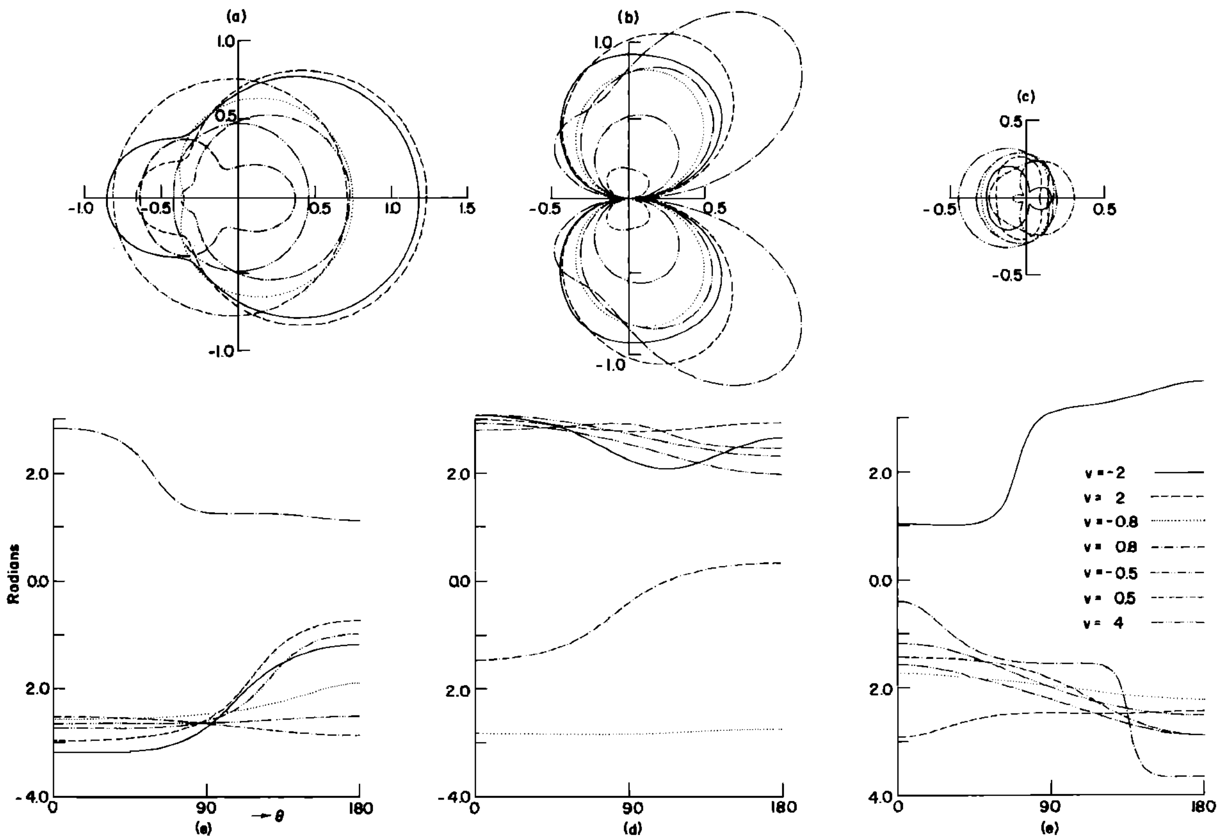

FIG. 6. Scattering amplitudes for a plane wave incident on a cylinder at $e=25°$, for various velocities: absolute value and phase, for various velocities. (a) Polar plot for $|g_p(\theta)|$; (b) polar plot for $|g_s^{(1)}(\theta)|$; (c) polar plot for $|g_s^{(2)}(\theta)|$; (d) phase of $g_p(\theta)$; (e) phase of $g_s^{(1)}(\theta)$; (f) phase of $g_s^{(2)}(\theta)$.

Figs. 3 and 4. Because of space considerations, the phase is also not shown. Such curves would show a jump in the phase at the above special points.

III. MOVING CYLINDER

Let us consider the case of a solid circular cylinder moving along its axis in an infinite medium (see Fig. 5). The incident wave is given by Eq. 1; the scattered waves are given by Eq. 2. In the present case, the integrals are recast in terms of cylindrical wave functions, yielding

$$\phi_0=\exp(ik_pz\cos e)\sum_{n=-\infty}^{\infty} i^nJ_n(k_pr\sin e)\exp(in\theta),$$

$$\psi_0=0,$$

$$\phi_1=\exp(ik_pz\cos e)\sum_n i^nA_nH_n(k_pr\sin e)\exp(in\theta),$$

$$\psi_1^{(i)}=\exp(ik_sz\cos f)\times\sum_n i^nB_n^{(i)}H_n(k_sr\sin f)\exp(in\theta),$$

$$\phi_2=\exp(iK_pz\cos e')\sum_n i^nA_n'J_n(K_pr\sin e')\exp(in\theta),$$

$$\psi_2^{(i)}=\exp(iK_sz\cos f')\times\sum_n i^nB_n'^{(i)}J_n(K_sr\sin f')\exp(in\theta),\quad i=1,2. \tag{7}$$

The subscripts 0, 1, and 2 denote the incident, scattered, and internal fields, respectively. The A_n, $B_n^{(i)}$, A_n', and $B_n'^{(i)}$ correspond to scattered compressional, scattered shear, internal compressional, and shear waves, respectively. The incident and internal fields are represented in terms of the nonsingular Bessel functions J_n, and the scattered waves are described by means of the Hankel functions of the first kind $H_n^{(1)}\equiv H_n$ [corresponding to the time factor $\exp(-i\omega t)$]. Inside the cylinder, the arguments $K_pr\sin e'$ and $K_sr\sin f'$ can become imaginary according to Fig. 2. In this case, the fields can be described by the modified Bessel functions I_n. The boundary conditions are the continuities of the displacements and stresses at $r=a$. The displacement vector $\mathbf{u}$ is given by

$$\mathbf{u}=\nabla\phi+\times\hat{\mathbf{z}}\nabla\psi^{(1)}+\nabla\times(\nabla\times\hat{z}\psi^{(2)}), \tag{8}$$

and the stresses can be derived from $\mathbf{u}$.[13] By applying

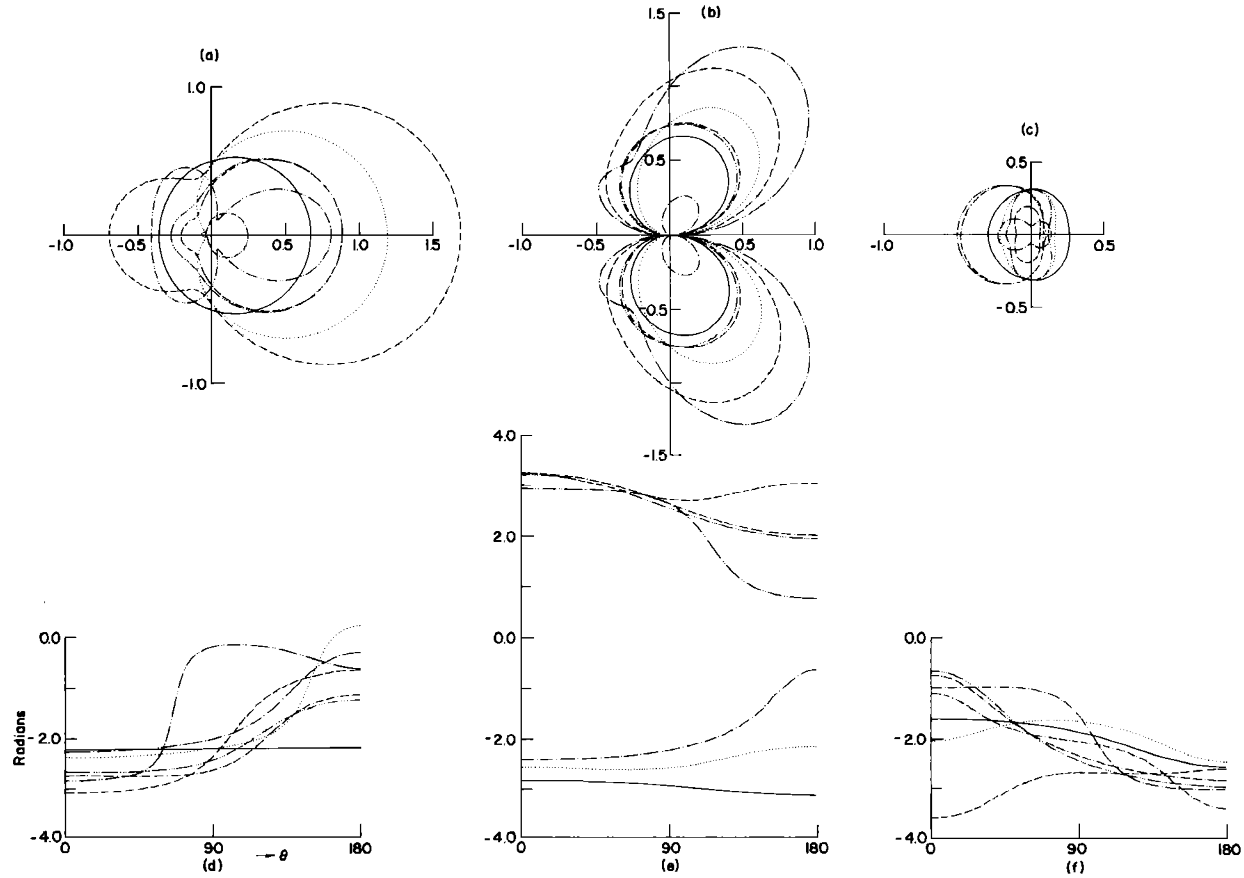

FIG. 7. Same as Fig. 6, with $e=45°$.

the boundary conditions and exploiting orthogonality properties of $\exp(in\theta)$ we get a set of six algebraic equations for the coefficients A_n, $B_n^{(1)}$, $B_n^{(2)}$, A_n', $B_n'^{(1)}$, and $B_n'^{(2)}$. In matrix form we have

$$\tilde{M}\cdot\mathbf{C}=\mathbf{R}, \tag{9}$$

where $\tilde{M}$ is a 6×6 matrix, $\mathbf{C}$ is the vector of the coefficients, and $\mathbf{R}$ is composed of the nonhomogeneous terms, determined by the incident wave. To bring out the intrinsic velocity effects, the two media are taken with identical properties in their proper frames of reference. Since Eq. 4 is valid for both the case of two half-spaces and the present one, all the conclusions of the previous section, such as criticality, hold here too. In Figs. 6–8, the scattering amplitudes $g_p(\theta)$ and $g_s^{(i)}(\theta)$ are given for various velocities and angles of incidence e. These functions describe the potentials ϕ_1, $\psi_1^{(i)}$ of Eqs. 7 at large distances, according to the asymptotic behavior

$$\phi_1\sim(2/i\pi k_p r\ \mathrm{sin}e)^{\frac{1}{2}}\exp(ik_p r\ \mathrm{sin}e)g_p(\theta),$$

$$\psi_1^{(i)}\sim(2/i\pi k_s r\ \mathrm{sin}f)^{\frac{1}{2}}\exp(ik_s r\ \mathrm{sin}f)g_s^{(i)}(\theta),$$

$$g_p(\theta)=\sum_{n=-\infty}^{\infty}A_n\exp(in\theta),$$

$$g_s^{(i)}(\theta)=\sum_{n=-\infty}^{\infty}B_n^{(i)}\exp(in\theta). \tag{10}$$

In the present results, we chose two identical media, as mentioned above, and $k_pa=1$. The three scattering amplitudes $g_p(\theta)$, $g_s^{(i)}(\theta)$, $i=1$, 2 provide essentially the most compact description of the elastic state. With these functions known, the displacement vector $\mathbf{u}$ can be derived (according to Eq. 8), and similarly the stress and strain tensors. Furthermore, the field at arbitrary points of the external region can be reconstructed from the knowledge of the field at large distances. Conversely, by exploiting orthogonality properties of vector spherical harmonics,[14] the scattering amplitudes can be recovered from measurements, e.g., of the displacement vector.

Inasmuch as the problem is symmetrical with respect to $\theta=0$ (the direction of incidence), the displacement vector satisfies $v(\theta)=v(-\theta)$. This prescribes for g_p, $g_s^{(2)}$

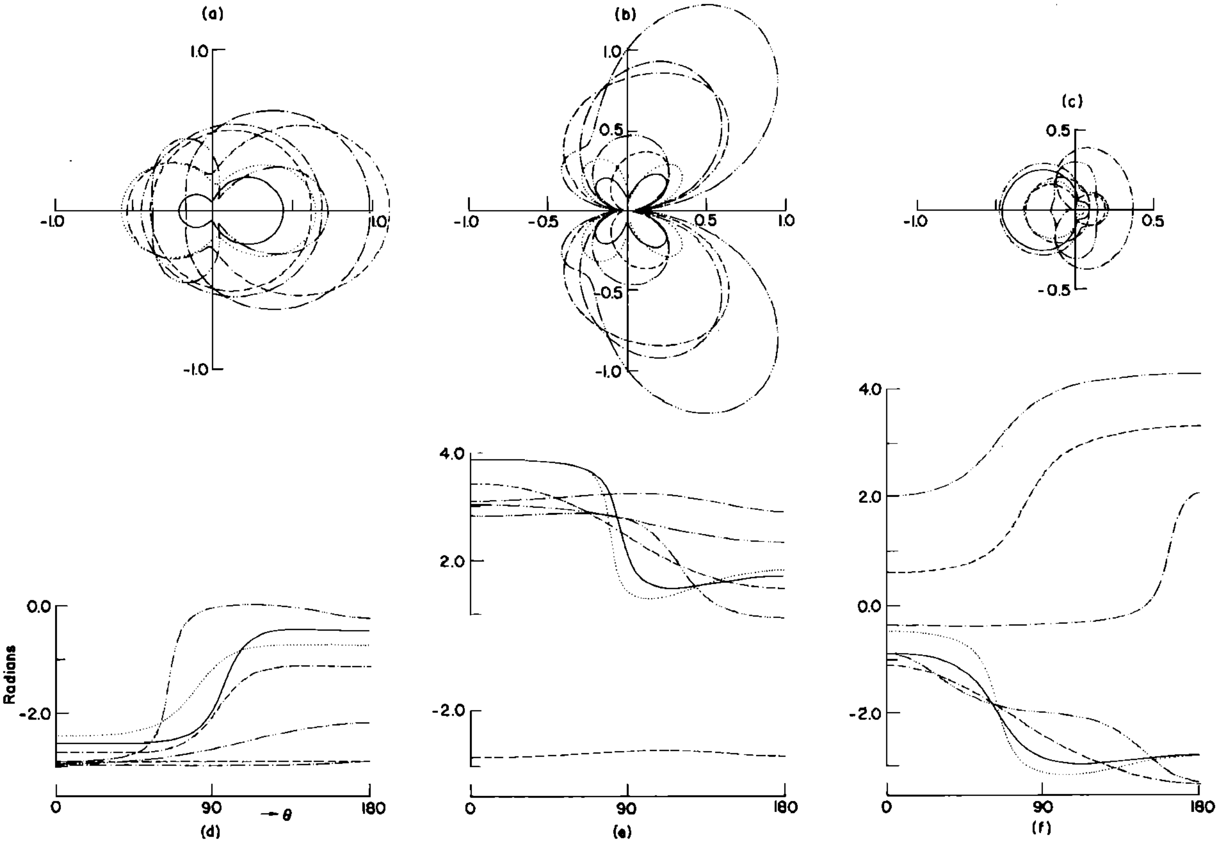

FIG. 8. Same as Fig. 6, with $e=65°$.

a symmetrical structure, and a skew-symmetric structure for $g_s^{(1)}$. This is seen in Figs. 6–8.

ACKNOWLEDGMENT

The computations connected with this paper were performed at the Computation Center of Tel-Aviv University.

* Now with the Department of Engineering Sciences, Tel-Aviv University, Ramat-Aviv, Israel.

[1] P. M. Morse and K. Uno Ingard, *Theoretical Acoustics* (McGraw–Hill, New York, 1968), Chap. 11.

[2] J. B. Keller, "Reflection and Transmission of Sound by a Moving Medium," J. Acoust. Soc. Amer. **27**, 1044–1047 (1955).

[3] J. W. Miles, "On the Reflection of Sound at an Interface of Relative Motion," J. Acoust. Soc. Amer. **29**, 226–228 (1957).

[4] H. S. Ribner, "Reflection, Transmission, and Amplification of Sound by a Moving Medium," J. Acoust. Soc. Amer. **29**, 435–441 (1957).

[5] C. Yeh, "Reflection and Transmission of Sound Waves by a Moving Fluid Layer," J. Acoust. Soc. Amer. **41**, 817–821 (1967).

[6] C. Yeh, "A Further Note on the Reflection and Transmission of Sound by a Moving Fluid Layer," J. Acoust. Soc. Amer. **43**, 1454–1455 (1968).

[7] E. W. Graham and B. B. Graham, "Effect of a Shear Layer on Plane Waves of Sound in a Fluid," J. Acoust. Soc. Amer. **46**, 169–175 (1968).

[8] C. Yeh, "Diffraction of Sound Waves by a Moving Fluid Cylinder," J. Acoust. Soc. Amer. **44**, 1216–1219 (1968).

[9] D. Censor, "Scattering of a Plane Wave at a Plane Interface Separating Two Moving Media," Radio Sci. **4**, 1079–1088 (1969).

[10] D. Censor, "Scattering of Electromagnetic Waves by a Cylinder Moving Along its Axis," IEEE Trans. Microwave Theory Tech. **MTT-17**, 154–158 (1969).

[11] D. Censor, J. Aboudi, and D. Neulander, "Reflection and Transmission of Elastic Waves by a Moving Slab," Appl. Sci. Res. (to be published).

[12] W. M. Ewing, W. S. Jardetzky, and F. Press, *Elastic Waves in Layered Media* (McGraw–Hill, New York, 1957), pp. 24 ff.

[13] See, e.g., H. Takeuchi, *Theory of the Earth's Interior* (Blaisdell, Waltham, Mass., 1966), pp. 23 ff.

[14] See, e.g., J. A. Stratton, *Electromagnetic Theory* (McGraw–Hill, New York, 1941), pp. 392 ff.