Harmonic and transient scattering from time varying obstacles

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Transient scattering as a tool for identification of objects and their physical properties through their natural (resonance) complex frequencies is now under intensive investigation by mathematicians and engineers. This promising method of inverse scattering is pertinent to mechanical as well as electromagnetic waves. Presently this problem is considered in the context of acoustics for time varying objects, such as pulsating cylinders and spheres. Together with a choice of a reasonably simple physical model and boundary conditions, this provides a framework for discussing a few canonical problems. Mathematically, the method employed here is essentially a perturbation technique. The representation in terms of compact symbolic differential operators which are manipulated algebraically helps to avoid a lot of cumbersome detail. The results display the creation of new spectral components and the associated new poles (i.e., new resonance frequencies) due to the time variation of the objects. The additional new features, when compared to the presumably known signature of the original object at rest, facilitate the identification of the details of the motion.

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INTRODUCTION

Scattering by time varying objects has been discussed, to some extent, in the past. This study is not concerned with simple problems of uniform motion leading to the well-known Doppler effect, which have been amply discussed in the past, e.g., see Gill,[1] and especially for electromagnetic waves see Censor,[2] and Van Bladel,[3] citing numerous references. Of special interest here are problems which cannot be solved by means of simple coordinate transformations. Problems of this kind have been discussed by Censor,[4] Cooper,[5] and Van Bladel and De Zutter,[6] for one-dimensional planar cases.[7] De Zutter[8] considers also oblique incidence. Scattering by expanding cylinders and spheres is considered by Censor,[9,10] who also discussed various modes of periodic motion perturbing cylinders and spheres.[11] Some additional references are mentioned by Censor[2] and Van Bladel.[3]

Presently, scattering of harmonic waves is considered for various motional modes perturbing the canonical cases of plane, cylindrical, and spherical scatterers. For simplicity the discussion is confined to acoustics, with the simple boundary condition prescribing the vanishing of the field on the surface (rigid objects). The technique is sufficiently general to be adapted to other boundary conditions (e.g., free surface) and different wave systems (e.g., electromagnetic waves). It must be noted, however, that in acoustics we need to include the simplifying assumption that the effects on the external medium are negligible. In electrodynamics in free space (vacuum) this problem does not arise. The refinement of the present model to include the fluid-dynamic problem of the motion induced in the external medium would make the problem considerably more complicated. The legitimacy of this assumption has been questioned previously by Rogers,[12] but it seems that as a first approximation it is adequate.

The method is described in a general manner first, and then applied to various examples of scattering from a vibrating and rippling plane. Normal and oblique incidence are considered. Similar problems of time varying cylinders and spheres are discussed.

The problem of transient scattering is the subject of intensive study at the moment. Most of the relevant literature deals with the electromagnetic problem. General reviews and introductions are given by Baum,[13–15] Dolph and Scott,[16] Bennett and Ross,[17] and Miller and Landt.[18] Transient scattering in velocity dependent systems, involving uniformly moving objects has been discussed recently by Censor.[19]

The impulse response for time varying scatterers is obtained from the response to harmonic excitation by contour integration. To this end the poles in the complex frequency plane must be identified. It is shown below that the motion creates new poles, or causes the migration of poles. The appearance of new pole configurations may contribute to the analysis of the motion of the object by means of its transient scattering response. In realistic situations the excitation will not be a unit impulse, which is a mere mathematical construction. Indeed, the object will be excited by a sharp pulse, and some additional signal processing will be necessary before the poles can be extracted from the scattered signal.

I. STATEMENT OF THE PROBLEM AND A GENERAL FORMALISM—HARMONIC EXCITATION

We are dealing with the problem of scattering from time varying objects. For the case of harmonic excitation the incident wave is chosen as

$$u_i(\mathbf{r},t) = e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{1}$$

a plane harmonic wave propagating in the direction of the unit vector $\hat{\mathbf{k}}$, with frequency ω and propagation vector $\mathbf{k} = \hat{\mathbf{k}}\omega/C$, with C denoting the phase speed of the medium. The scattered wave $u_s(\mathbf{r},t)$ is a solution of the wave equation, such that the sum $(u_i + u_s)$ satisfies the boundary conditions on the surface of the object, and at large distances u_s behaves as an outgoing wave, satisfying the pertaining radiation condition. For time invariant scatterers the boundary conditions and (1) prescribe a time harmonic solution, such that the scattered wave is of the form $u_{s\omega}(\mathbf{r})e^{-i\omega t}$, where $u_{s\omega}$ is a function of ω, and written in this notation to distinguish it from u_s. In the presence of time varying surfaces the spectral content of the scattered wave differs from that of the exciting wave, hence, in general

$$u_s(\mathbf{r},t) = \int_{-\infty}^{\infty} d\nu\, A(\nu) u_{s\nu}(\mathbf{r}) e^{-i\nu t}, \qquad (2)$$

where $A(\nu)$ denotes the amplitude of the appropriate spectral component. A time invariant surface is generally defined by the implicit function $F_0(\boldsymbol{\rho}) = 0$, where $\boldsymbol{\rho}$ is the position vector locating the surface. For a time dependent surface we have in general

$$F(\boldsymbol{\rho},t) = 0. \qquad (3)$$

The boundary condition is prescribed on the surface $\mathbf{r} = \boldsymbol{\rho}$, subject to (3), and for simplicity, the boundary condition used throughout the present study is

$$u_i(\boldsymbol{\rho},t) + u_s(\boldsymbol{\rho},t) = 0, \qquad (4)$$

although other boundary conditions are applicable too. The general problem is to compute (2) subject to (3) and (4), with (2) being an appropriate solution of the wave equation, as explained above.

Inasmuch as a general method is not available, we confine the discussion to cases where the time varying surfaces are perturbed fixed shapes, e.g., time dependent perturbed planes, cylinders, spheres, and $\boldsymbol{\rho}$ is given in the form

$$\boldsymbol{\rho} = \boldsymbol{\rho}_0 + \boldsymbol{\rho}_1(t). \qquad (5)$$

In this specialized form (4) now reads

$$\exp[i\mathbf{k}\cdot\boldsymbol{\rho}_0 + i\mathbf{k}\cdot\boldsymbol{\rho}_1(t) - i\omega t] + \int_{-\infty}^{\infty} A(\nu) u_{s\nu}[\boldsymbol{\rho}_0 + \boldsymbol{\rho}_1(t)] \exp(-i\nu t)\, d\nu = 0. \qquad (6)$$

The problem reduces to finding such constant coefficients $A(\nu)$ which satisfy (6) for all t.

The spectral contents of the incident wave at the perturbed surface is given by the Fourier transform

$$u_i(\boldsymbol{\rho},t) = \exp(i\mathbf{k}\cdot\boldsymbol{\rho}_0 - i\omega t)\int_{-\infty}^{\infty} d\mu\, f(\mathbf{k},\mu)\exp(-i\mu t) = \exp(i\mathbf{k}\cdot\boldsymbol{\rho}_0)\int_{-\infty}^{\infty} d\mu\, f(\mathbf{k},\mu-\omega)\exp(-i\mu t). \qquad (7)$$

In order for (6) to be satisfied for all t, we have to look for the corresponding spectral components in the scattered wave. However, there is no point in formally recasting $u_s(\boldsymbol{\rho},t)$ or $u_{s\nu}(\boldsymbol{\rho})$ in a Fourier integral, because these functions are unknown. On the other hand, $u_{s\nu}(\boldsymbol{\rho}_0)$ is available in closed form for numerous scattering problems involving time invariant surfaces. Therefore the present approach will be to expand $u_{s\nu}[\boldsymbol{\rho}_0 + \boldsymbol{\rho}_1(t)]$ in a Taylor series about $u_{s\nu}(\boldsymbol{\rho}_0)$, and for practical problems the series can be truncated if the perturbation is small. In a compact notation the Taylor series is represented in the symbolic form

$$u_{s\nu}[\boldsymbol{\rho}_0 + \boldsymbol{\rho}_1(t)] = \exp[\boldsymbol{\rho}_1(t)\cdot\nabla] u_{s\nu}(\boldsymbol{\rho}_0), \qquad (8)$$

by which it is understood that the exponential has to be formally expanded in powers of $(\boldsymbol{\rho}_1\cdot\nabla)$, where ∇ is the gradient operator, and the differential operators thus obtained applied to $u_{s\nu}(\boldsymbol{\rho}_0)$. The advantage of this notation is in the fact that as far as t is concerned, ∇ can be manipulated algebraically. Similarly to (7) we now write

$$\exp[\boldsymbol{\rho}_1(t)\cdot\nabla] = \int_{-\infty}^{\infty} d\sigma\, g(\nabla,\sigma)\exp(-i\sigma t) \qquad (9)$$

and exploiting the convolution theorem, (6) becomes

$$f(\mathbf{k},\mu-\omega)\exp(i\mathbf{k}\cdot\boldsymbol{\rho}_0) = -\int_{-\infty}^{\infty} d\nu\, A(\nu) g(\nabla,\mu-\nu) u_{s\nu}(\boldsymbol{\rho}_0). \qquad (10)$$

Provided the functions shown in (10) actually exist and can be found, the problem now reduces to deconvolving (10) and finding $A(\nu)$, which are necessary to specify the scattered wave (2).

The general method described above will be applied to specific examples of interest in the following section. It is noteworthy that this formalism, in a more specialized way, is exactly what people have done in the past, e.g., see De Zutter.[8] Here it is elevated to the position of a general formalism, including differential operators.

II. TRANSIENT SCATTERING—GENERAL CONSIDERATIONS

To complete the general introduction, the question of transient scattering and the related subject of the appropriate pole configuration is introduced. However, the reader might prefer to skip this section in a first reading and go on to examples of perturbed planes scattering harmonic waves.

Transient scattering is at the present investigated very intensively by mathematicians and engineers, especially in the field of electromagnetic wave scattering. From the impulse response a map of poles can be obtained in the complex frequency plane, constituting a signature which characterizes the scatterer in question. This, therefore, provides an inverse scattering method which facilitates the identification of scatterers and certain properties thereof. The special contribution of the present study, as given in subsequent sections, is the analysis of the effect of time variation of the scatterer on the scattered wave, and the interpretation in terms of the associated pole configuration. Since the present study brings together (for the first time, it is believed) two remote subjects, namely, transient scattering analysis and scattering in the presence of time varying boundaries, a short introduction seems to be in order.

Physically speaking, the impulse response is what is measured when a bell is given a brisk tap, for example. The bell responds with a ringing (the impulse response scattered wave) according to its natural resonance frequencies. These depend on the bell's shape and materials involved. Since the sound is transmitted into the environment, its amplitude decays as the energy departed to the bell is slowly dissipated. Hence, we are dealing with complex frequencies describing the exponential decay. A map of these frequencies in the complex plane characterizes the object in question and provides a signature. In the following mathematical introduction it will be clarified how the impulse response of a time varying object may contribute to studying its motional characteristics from the changes in its pole map, compared to the unperturbed object.

For simplicity the excitation is taken as a plane-wave unit impulse

$$\delta\left(t-\frac{\hat{\mathbf{k}}\cdot\mathbf{r}}{C}\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\,d\omega\,. \tag{11}$$

Consequently, the impulse response is found by integrating the response to an incident harmonic plane wave over all frequencies. In (2) the response of a time varying object to an incident wave of frequency ω is indicated. The dependence on ω has been suppressed in the notation (2), but if we write explicitly $u_s(\omega,\mathbf{r},t)$ to denote the frequency of the incident wave, then the impulse response will be given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}u_s(\omega,\mathbf{r},t)d\omega=\sum\text{residues}\,. \tag{12}$$

From (12) it is clear that if the residues are computed the complicated integration along the real axis can be avoided. Actually the poles are located in the lower half of the complex ω plane. The question of the multiplicity of the poles, and the extraction of poles from the impulse response is discussed in Baum,[13–15] Dolph and Scott,[16] Bennett and Ross,[17] and Miller and Landt,[18] and the references cited by them. The impulse response will take the form

$$u(\mathbf{r},t)=\sum_n W_n(k_n\mathbf{r})\exp(-i\omega_n t) \tag{13}$$

for objects possessing simple poles, where ω_n are the poles in the complex ω plane and W_n are solutions of the associated Helmholtz equation, with $k_n=\omega_n/C$.

Our task in connection with the analysis of transient scattering from time varying objects is to identify the new poles introduced by motional effects and their multiplicity.

III. THE PLANE SCATTERER

The time varying perturbed plane scatterer has been considered by numerous authors, see Censor,[2] and Van Bladel[3] for some references. An exact solution, based on the D'Alembert solution of the one-dimensional wave equation has been given by Censor[4] (see also Van Bladel and De Zutter,[6] and the comment by Censor with reply of authors,[7] and De Zutter[8]).

The one-dimensional case of a harmonically vibrating mirror is of importance because of its relative simplicity, which serves as an example for the general formalism discussed above, and as an introduction to more complicated cases discussed below. The incident wave is given by

$$u_i=\exp(ikx-i\omega t) \tag{14}$$

and the reflected wave is first written in a general form

$$u_s=\int A(\nu)\exp(-ik_\nu x-i\nu t)\,d\nu\,,\quad k_\nu=\frac{\omega}{C}, \tag{15}$$

which defines outgoing (i.e., reflected) waves. It must be stressed, and this point will be picked up below, that the choice of the scattered wave and the legitimacy of this choice is the most heuristic step in the whole treatment. The motion is described by

$$\rho=x_0+\xi\sin\Omega t\,. \tag{16}$$

Hence the incident wave (14) and the reflected wave (15) at the position of the scatterer (16) are given by

$$\exp(ikx_0+i\xi\sin\Omega t-i\omega t)=\exp(ikx_0-i\omega t)\sum_m J_m(k\xi)\exp(im\Omega t)\,,$$

$$\int A(\nu)\exp(-ik_\nu x_0)\sum_n(-1)^n J_n(k_\nu\xi)\times\exp[-i(\nu-n\Omega)t]\,, \tag{17}$$

respectively. In order that the boundary condition (4) be satisfied for all t, ν in (17) must be quantized according to $\nu=\omega-q\Omega$, $q=0,\pm 1,\pm 2,\ldots$, so that the incident and reflected waves contain the same spectral components, therefore the reflected wave is given by

$$\exp(-i\omega t-ikx_0)\sum_n\sum_q A_q(-1)^n J_n\left(\frac{\omega-q\Omega}{C}\xi\right)\times\exp[(q+n)\Omega t]\,, \tag{18}$$

and the problem reduces to the computation of A_q. For simplicity, without affecting the generality of the problem, assume that the plane vibrates about $x_0=0$. For each mode m we selected from (18) the values of n, q, satisfying $m=n+q$, hence the equation on A_q becomes, as a special case of (10),

$$J_m(k\xi)+\sum_q A_q(-1)^{m-q}J_{m-q}\left(\frac{\omega-q\Omega}{C}\xi\right)=0\,. \tag{19}$$

This is an infinite system of linear algebraic equations, and can be solved approximately by truncating the range of m, q. Trivially, $q=m=0$ yields $A_0=-1$ and $\nu=\omega$ as the frequency of the reflected wave. If we allow $q=m=0,\pm 1$, we get three equations on the unknowns A_q, $q=0,\pm 1$,

$$\begin{aligned}&J_0(k\xi)+A_0J_0(k\xi)-A_1J_{-1}\{[(\omega-\Omega)/C]\xi\}\\&\quad-A_{-1}J_1\{[(\omega+\Omega)/C]\xi\}=0\,,\\&J_1(k\xi)-A_0J_1(k\xi)+A_1J_0\{[(\omega-\Omega)/C]\xi\}\\&\quad+A_{-1}J_2\{[(\omega+\Omega)/C]\xi\}=0\,,\\&J_{-1}(k\xi)-A_0J_{-1}(k\xi)+A_1J_{-2}\{[(\omega-\Omega)/C]\xi\}\\&\quad+A_{-1}J_0\{[(\omega+\Omega)/C]\xi\}=0\,.\end{aligned} \tag{20}$$

There is no point in deriving detailed analytical approximations for high order q, m values. What is more interesting are the general properties of such solutions and the effects on the pole location. Evidently A_q will be expressed as rational functions in terms of Bessel functions, i.e., a ratio of linear combinations of Bessel functions of various (real) integral orders and arguments $[(\omega-\alpha\Omega)/C]\xi$ for various integral α's. The poles of A_q are therefore available for computation. In general, the scattered wave will be of the form

$$u_s=\sum_q A_q\exp\left[-i(\omega-q\Omega)\left(t+\frac{x}{C}\right)\right]. \tag{21}$$

The explicit computation of the location of the poles is out of the scope of the present study and has to be found numerically. By correlation of numerical results for various ξ and Ω/ω values with experimental data, in principle it is possible to identify these characteristic parameters of the motion from the known pole configurations.

De Zutter[8] discussed the case of oblique incidence for scattering of a harmonic electromagnetic wave from a harmonically vibrating plane. It is important to review this example in the acoustical context, because fundamental properties of the solution are a result of the kinematics involved and are not a result of discussing electromagnetic waves or using an exact special relativistic formalism. The incident wave is given by

$$u_i = \exp(ik_x x + ik_y y - i\omega t), \quad k_x^2 + k_y^2 = \omega^2/C^2, \quad (22)$$

and its value on $x = \xi \sin \Omega t$ is

$$\exp(ik_y - i\omega t) \sum_m J_m(k_x \xi) \exp(im\Omega t). \quad (23)$$

The general form of the scattered wave is chosen as

$$u_s = \int A(\nu) \exp(-ik_{\nu x} x + ik_{\nu y} y - i\nu t)\, d\nu, \quad (24)$$

$$k_{\nu x}^2 + k_{\nu y}^2 = \nu^2/C^2,$$

constituting a superposition of reflected waves for which the proper amplitudes and $k_{\nu x}$, ν must be determined. At $x = \xi \sin \Omega t$ we obtain for the scattered field at the boundary

$$\int A(\nu) \exp(ik_{\nu y} y - i\omega t) \sum_n (-1)^n J_n(k_{\nu x}\xi) \exp(in\Omega t),$$

$$k_{\nu y} = k_y. \quad (25)$$

Hence again (23), (25), and (4) prescribe $\nu = \omega - q\Omega$ and (24) becomes

$$u_s = \sum_q A_q \exp[-ik_{qx} x - ik_y y - i(\omega - q\Omega)t],$$

$$k_{qx} = [k_y^2 - (\omega - q\Omega)^2/C^2]^{1/2}. \quad (26)$$

Although we are dealing with simple scalar (as opposed to vector electromagnetic) waves, and no special relativistic considerations are involved, (26) predicts an angular spectrum of scattered plane waves. This includes, therefore, the aberration phenomenon which is sometimes erroneously considered to be a relativistic effect. Another point which appears here but has been previously noted by De Zutter[8] is in fact that (26) predicts evanescent waves. This result hinges on the choice of (24) as a superposition of reflected waves. The legitimacy of using (24) in this form is however open to discussion. Should we have included in the scattered field also waves with $+ik_{\nu x}x$ in the exponential (24)? Such waves with the appropriate branch of (26) could cancel the evanescent waves obtained by De Zutter.[8] Or, if evanescent waves are present, does the choice of (24) guarantee all such waves that must be present on the surface? These questions are related to the representation of the scattered wave in two and three dimensions. It has been shown, e.g., see Twersky[20–22] that the representation of the scattered wave in terms of outgoing waves is legitimate outside the circle (for two dimensions) or sphere (in three dimensions) enclosing the scatterer. In fact, a somewhat weaker condition is stated,[20–23] but this is unessential for the present argument. Indeed, if the surface of a two- or three-dimensional object is indented in such a way that a leaky cavity is formed, it is clear on grounds of a physical argument that standing waves can be present, i.e., outgoing waves alone are not sufficient to describe the situation. The answer to the question of choosing the scattered wave in terms of outgoing waves will have to be resolved in the future, at the moment the present author is not aware of an existing answer.

The method of solution for A_q in (26) is similar to the normal incidence case, in that we have an infinite system of equations

$$J_m(k_x\xi) + \sum_q A_q(-1)^{m-q} J_{m-q}(k_{qx}\xi) = 0. \quad (27)$$

Once the harmonic excitation solution (26) is available, the impulse response is derived by (12), and the pole configuration depends on the parameters of the incident wave ω,**k** and the parameters of the motion, i.e., Ω,ξ.

The problems discussed thus far have been considered in the literature before and served here to demonstrate the general formalism which deals with much more intricate problems as well. An obvious extension of the above problems is the periodic motion replacing (16) by

$$\rho = \sum_\gamma \xi_\gamma \sin(\gamma\Omega t + \phi_\gamma), \quad (28)$$

whereby (14) becomes on the scatterer

$$\exp(ik\rho - i\omega t) = \prod_\gamma \Big(\sum_{m_\gamma} J_{m_\gamma}(k\xi_\gamma) \times \exp[im_\gamma(\gamma\Omega t + \phi_\gamma) - i\omega t] \Big). \quad (29)$$

Obviously, a truncation of γ, m_γ is necessary to derive explicit approximations. The reflected wave is given by (21) but the bookkeeping of terms from (29) and the analog of (25) is getting cumbersome. In the absence of an obvious application to a specific problem of interest, this argument will not be further pursued.

A much more practical problem is presented by scattering from a time varying corrugated, or rippling plane. To this class of problems belongs scattering of sound from the surface of the sea, both for acoustic propagation above the surface or underwater. In all cases our model is simplified by neglecting the effect of the motion of the media caused by the moving surface. This class of problems is also related to scattering of light and sound by surface waves. In general we are dealing here with a surface described by

$$\rho = x_0 + \rho_1(y,z,t). \quad (30)$$

More specifically, we shall consider the problem of a one-dimensional mechanical wave on the surface according to

$$\rho = \xi \sin(Ky - \Omega t), \quad K = \Omega/\zeta, \quad (31)$$

where ζ is the phase velocity of the mechanical wave. This mode (30) describes, in a sense, a moving diffraction grating. Upon replacing (31) by a standing wave

$$\rho = \xi \sin Ky \sin \Omega t, \quad (32)$$

we encounter a time varying grating occupying the same position, as defined by its modes, i.e., the zeros of $\sin Ky$. According to (31), the incident wave (14) and the reflected wave (24) on the surface become

$$\exp(ik\rho - i\omega t) = \sum_m J_m(k\xi)\exp[im(Ky - \Omega t) - i\omega t],$$

$$\int A(\nu)\sum_n(-1)^n J_n(k_{\nu x}\xi) \times\exp[in(Ky - \Omega t) + ik_{\nu y}y - i\nu t]\,d\nu, \tag{33}$$

respectively. The temporal structure of (33) again prescribes that the scattered wave can be quantized according to $\nu = \omega + q\Omega$, but here we have to also accommodate the spatial dependence on y, hence (24) instead of (15) must be used. The condition on the surface now becomes

$$\sum_m J_m(k\xi)\exp[im(Ky - \Omega t)] + \sum_q A_q \sum_n\{(-1)^n J_n(k_{qx}\xi) \times\exp[i(nK + k_{qy})y]\exp[-i(n+q)\Omega t]\} = 0. \tag{34}$$

By equating terms $m = n + q$ (34) holds for all t, and by further defining $k_{qy} = qK$, the condition (34) is satisfied for all y. But this prescribes

$$k_{qx}^2 = (\omega + m\Omega)^2/C^2 - (qK)^2 \tag{35}$$

and the system of equations on A_q becomes (27), with k_x replaced by k and k_{qx} given by (35). Again the problem is characterized by an angular spectrum of scattered plane waves propagating in various directions, also (35) is conducive to evanescent waves and the same comments made above in connection with the oblique incidence case apply here. The motion induced poles for the impulse response associated with the last problem now depend on another parameter, namely ζ, the phase velocity characterizing the system supporting the surface waves. At least in principle, this property can be studied by means of the scattering products, of the harmonic or the impulse response. The last problem can be combined with oblique incidence, or periodic excitation or both. Problems involving spatial combination of waves may be conceived. Other spatial wave modes such as cylindrical waves on the surface can be studied. The formalism is sufficiently general to encompass all such cases. However, in a theoretical study as we have here, such detailed discussions are not justified.

IV. THE CIRCULAR CYLINDER

Many problems of interest involving surfaces which, in the unperturbed state have a radius of curvature large compared to wavelength, can be tackled with the methods described in the previous section. On the other hand, when characteristic parameters, e.g., radius, are comparable to the wavelength, the curvature of the unperturbed surface must be taken into account. The circular cylinder provides a convenient canonical example for demonstrating the effect of time varying surfaces. Problems of this kind have been considered previously by Censor,[2,9–11] citing other literature references (see also Van Bladel[3]).

The novel feature, compared to problems of the plane scatterer, is the appearance of the differential operators introduced in the general formalism (10). The unified approach to examples given here facilitates the discussion of other classes of problems which may be of interest for engineering applications. The response to harmonic and transient scattering is considered, as a method of studying the motion via the scattered wave.

Before proceeding to the main line of this section, it seems advisable to try and understand which problems will be simple to handle. As it happens so often, problems are amenable to straightforward analytical treatment if they can be expressed in terms of the components of the coordinate system in question in a simple way. This is the reason for the relative simplicity of the analysis of the plane scatterer problems considered above. In order not to be too vague, consider for example, the case of a circular cylinder oriented along the z axis and vibrating along the y axis, with the incident wave given by (14). The position of the surface of the cylinder is defined by

$$\boldsymbol{\rho} = \hat{\mathbf{x}}\,a\cos\phi + \hat{\mathbf{y}}\,(a\sin\phi + \xi\cos\Omega t), \tag{36}$$

i.e., this is a cylinder of radius a whose center is given by the coordinates $x = 0$, $y = \xi\cos\Omega t$. The incident wave (14) is first written in terms of polar coordinates $x = r\cos\phi$, $y = r\cos\phi$, and then according to (36) we have on the surface of the cylinder the incident wave in the form

$$\exp\{ik\,[a^2\cos^2\phi + (a\sin\phi + \xi\cos\Omega t)^2]^{1/2}\cos\phi - i\omega t\}. \tag{37}$$

Although we can, in principle, recast (37) in a series displaying the spectrum in ϕ and t, and finally derive a form of the type (10), this is certainly a complicated problem. To bring up the essential issues involved in scattering by time varying objects, and until a need for a specific problem does arise, we shall confine ourselves to simple "canonical" problems.

A relatively simple problem demonstrating the method of solution and its limitation, is provided by the pulsating cylinder, defined by a harmonically varying radius,

$$\rho = a + \xi\sin\Omega t. \tag{38}$$

The incident wave (14) at the surface becomes

$$\exp(ik\rho\cos\phi - i\omega t) = \sum_m i^m J_m(ka + k\xi\sin\Omega t)\exp(im\phi - i\omega t). \tag{39}$$

We have the option of finding the spectrum of (38) by a Fourier transformation, but practical reasons suggest that for ξ sufficiently small with respect to a, (38) be recast in a Taylor series. In practice this series will be approximated by a finite number of terms, up to the desired power of $k\xi$. We therefore rewrite (38) as

$$\sum_m i^m\exp[(k\xi\sin\Omega t)\partial]J_m(ka)\exp(im\phi - i\omega t) = \sum_m\sum_n J_n(-ik\xi\partial)J_m(ka)i^m \times\exp(im\phi)\exp[-i(\omega - n\Omega)t], \tag{40}$$

where ∂ symbolizes a differentiation of J_m with respect to the argument, and $J_n(-ik\xi\partial)$ is understood as a representation of J_n in a power series, where the power of ∂ signifies the number of times this operator is applied. Of course, we have to use $J_{-n} = (-1)^n J_n$ in order to have positive powers of ∂. One can also use a representation in terms of modified Bessel functions for J_n of imaginary argument. The scattered wave is chosen in terms of the Hankel function of the first kind, which in conjunction with $e^{-i\nu t}$, in the form

$$u_s = \sum_m \int i^m A_m(\nu) H_m(k_\nu r) \exp(im\phi - i\nu t)\, d\nu\,, \tag{41}$$

where $k_\nu = \nu/C$, and because of the discrete spectrum (39), this becomes

$$u_s = \sum_m \sum_q i^n A_{qm} H_m(k_q r) \exp[im\phi - i(\omega - q\Omega)t]\,. \tag{42}$$

On the surface we have, similar to (40)

$$H_m(k_q a + k_q \xi \sin \Omega t) = \exp(k_q \xi \sin \Omega t \partial) H_m(k_q q) = \sum_l \exp(il\Omega t) J_l(-ik_q \xi \partial) H_m(k_q a) \tag{43}$$

and the sum of (40) and (42) subject to (43) vanishes according to the boundary condition (4). Exploiting the orthogonality of the exponentials, and canceling $e^{-i\omega t}$, the system of equations for the coefficients is found in the form

$$J_n(-ik\xi\partial) J_m(ka) + \sum_q A_{qm}\, J_{n-q}(-ik_q \xi \partial) \times H_m(k_q a) = 0\,, \quad k_q = (\omega - q\Omega)/C\,. \tag{44}$$

Explicit approximations to the problems of the pulsating cylinder as well as other problems, some of them reviewed here, are given by Censor,[11] but here a systematic approach and a compact notation facilitates the discussion of such problems without going into cumbersome detail. It is evident from (44) that A_{qm} involves ratios of J_m, H_m functions, and derivatives thereof. For transient scattering this means that new complex poles appear in the complex ω plane. The subject of deriving A_{qm} in (44) and specifying the locations of the complex poles, belongs to a formidable numerical project that cannot be included in the framework of the present theoretical investigation.

The above problem of the pulsating cylinder corresponds to the vibrating plane. The problem demonstrated above for a plane supporting displacement waves corresponds to a cylinder supporting displacement waves along the generator and circumferential displacement waves. The first mode is characterized by a radius varying according to

$$\rho = a + \xi \sin(Kz - \Omega t)\,. \tag{45}$$

Skipping some intermediate steps, this leads to the incident wave (14) becoming on the surface, subject to (45),

$$\sum_m \sum_n J_n(-ik\xi\partial) J_m(ka) \times i^m \exp[im\phi + in(Kz - \Omega t) - i\omega t]\,, \tag{46}$$

very similar to (40), except that in (46) we have dependence on z. Therefore instead of (42) we now have

$$u_s = \sum_m \sum_q i^m A_{qm} H_m(k_{q\perp} r) \times \exp[ik_{qz} + im\phi - i(\omega + q\Omega)t]\,,$$
$$k_{qz}^2 + k_{q\perp}^2 = (\omega + q\Omega)^2/C^2\,. \tag{47}$$

On the surface (45), u_s becomes

$$\sum_m \sum_q \sum_l (J_l(-ik_{q\perp}\xi\partial) H_m(k_{q\perp} a) A_{qm} i^m \times \exp(im\phi + ik_{qz} z) \exp\{-i[\omega + (q + l\Omega)]t + ilKz\})\,. \tag{48}$$

Exploiting the orthogonal properties we finally find the system of equations for computing the coefficients

$$J_n(-ik\xi\partial) J_m(ka) + \sum_q J_{n-q}(-ik_{q\perp}\xi\partial) H_m(k_{q\perp} a) A_{qm} = 0\,, \tag{49}$$

where we have used $n = q + l$ and $k_{qz} = -qK$, and obtained a result very similar to (44).

The new feature introduced by circumferential surface perturbation waves is the effect of time variation on the ϕ dependence. Typical modes are considered by Censor.[11] Choosing as an example

$$\rho = a + \xi \sin \phi \cos \Omega t, \tag{50}$$

we have a mode of motion which for very small ξ/a approximates (37), i.e., the lineal vibration. The incident wave (14) on the surface becomes

$$\sum_m i^m \exp(k\xi\partial \sin \phi \cos \Omega t) J_m(ka) \exp(im\phi - i\omega t)\,. \tag{51}$$

By recasting

$$2 \sin \phi \cos \Omega t = \sin(\phi + \Omega t) + \sin(\phi - \Omega t)$$

we obtain

$$\sum_m \sum_n \sum_{n'} \left[J_n\left(\frac{-ik\xi\partial}{2}\right) J_{n'}\left(\frac{-ik\xi\partial}{2}\right) J_m(ka) \times \exp[i(n - n')\Omega t - i\omega t] \exp[i(n + n' + m)\phi] \right]. \tag{52}$$

The scattered wave is therefore chosen as

$$\sum_\mu \sum_q A_{q\mu}\, i^\mu H_\mu(k_q r) \exp[i\mu\phi - i(\omega + q\Omega)t]\,, \tag{53}$$

and subject to (50), on the surface yields

$$\sum_\mu \sum_q \sum_u \sum_{u'} \left[J_u\left(\frac{-ik_q\xi\partial}{2}\right) J_{u'}\left(\frac{-ik_q\xi\partial}{2}\right) H_\mu(k_q a) \times \exp[i(\mu + u + u')\phi] \times \exp[-i(\omega + q\Omega + u'\Omega - u\Omega)] t i^\mu A_{q\mu} \right]. \tag{54}$$

The sum of (52) and (54) vanishes on the surface for all ϕ, t hence by the orthogonality of the exponentials we have $n - n' = u - u' - q$, $n + n' + m = u + u' + \mu$. This leads to

$$J_n(-ik\xi\partial/2) J_{n'}(-ik\xi\partial/2) J_m(ka) + \sum_u \sum_{u'} i^\mu A_{q\mu} J_u\left(\frac{-ik_q\xi\partial}{2}\right) \times J_{u'}\left(\frac{-ik_q\xi\partial}{2}\right) H_\mu(k_q a) = 0\,,$$
$$\mu = n + n' + m - u - u'\,, \quad q = n - n' - u + u'\,. \tag{55}$$

This is as complicated a problem as we wish to display in the present discussion, and serves to show that although the general formalism (10) can be used, the details, even for an innocent looking mode of motion like (50), can be very cumbersome.

An entirely different problem is offered by an expanding cylinder, whose radius changes according to

$$\rho = a + vt\,, \quad a,v = \text{const}\,. \tag{56}$$

Relevant literature about expanding objects is cited by Censor.[2] This problem, which in one (Cartesian) dimension simply defines the Doppler effect, is very complicated when it comes to radial motion. The incident wave (14) becomes on the surface

$$\exp[ik(a+vt)\cos\phi - i\omega t] = \sum_m J_m(ka + kvt)\exp(im\phi - i\omega t)\,. \tag{57}$$

The Taylor expansion yields

$$\sum_m i^m \exp(kvt\partial)\, J_m(ka)\exp(im\phi - i\omega t)\,, \tag{58}$$

but this expansion in powers of kvt constitutes a series of secular terms [i.e., if only a finite number of terms in the expansion is taken and t increases, then the sum (58) grows to infinity and does not converge to (57)]. On the other hand, a claim that our approximation applies to small t is inconsistent with the fact that in harmonic scattering the results should be applicable to all t. This paradox can be dealt with in the context of transient scattering. Although (13) is derived from an integration of harmonic modes over all real frequencies, it consists of exponentially decaying terms, according to the location of the pole ω_n in the complex ω plane. Hence, even though our solution will involve powers of kvt, for large values of t the exponential decay takes over and suppresses the growth of t^h, where h is some integral positive power. For a scattered field

$$\sum_m \int A_m(\nu) i^m H_m(k_\nu r)\exp(im\phi - i\nu t)\, d\nu\,, \tag{59}$$

we obtain on the surface (56) the condition

$$\exp(kvt\partial)\, J_m(ka)\exp(-i\omega t) + \int A(\nu)\exp(k_\nu vt\partial)\, H_m(k_\nu a)\exp(-i\nu t)\, d\nu = 0, \tag{60}$$

which must hold for all t, at least within the interval for which we derive an adequate approximation. To the first order of t, which is also the order of the approximation in terms of the Mach number v/C, (60) reads

$$J_m(ka)\exp[-i\omega t + kvt\partial\, J_m(ka)/J_m(ka)] + \int A(\nu) H_m(k_\nu a)\exp\left(-i\nu t + \frac{kvt\partial H_m(ka)}{H_m(ka)}\right) d\nu\,, \tag{61}$$

where we have approximated

$$1 + kvt\partial\, J_m(ka)/J_m(ka) = \exp[kvt\partial\, J_m(ka)/J_m(ka)] + O(kvt)^2\,,$$

etc. For (61) to be satisfied for all t, we choose

$$\nu_m = \omega + ikv\left(\frac{\partial J_m(ka)}{J_m(ka)} - \frac{\partial H_m(ka)}{H_m(ka)}\right), \tag{62}$$

and

$$A_m(\nu_m) = -J_m(ka)/H_m(k_{\nu_m}a)\,, \quad k_\nu = \nu/C\,, \tag{63}$$

where ν_m is given by (62). Hence the integral (59) reduces to one term as given by (63). This is essentially the result obtained before by Censor.[10] It is interesting to note that the response to the harmonic incident wave gives rise to a complex frequency. The location of (62) for real ω, k, and v is complex in the ν_m frequency plans. Inasmuch as k_{ν_m} in the denominator (63) is different from ω/C of the time invariant cylinder, pole migration is evident. The new locations of the migrating poles, for such complex values of ν (62) which cause $H_m(k_{\nu_m}a)$ (63) to vanish are indicative of the motional effects and serve to detect the new characteristics from the impulse response according to (12). In order to derive higher order in kvt approximations for the scattered wave, we use the first-order approximation

$$\sum_m i^m \frac{-J_m(ka)}{H_m(k_{\nu_m}a)} H_m(k_{\nu_m} r)\exp(im\phi - i\nu_m t) \tag{64}$$

and subtract its value on the surface from the incident wave, writing

$$e^{kvt\partial} J_m(ka) e^{-i\omega t} - \frac{J_m(ka)}{H_m(k_{\nu_m}a)} \times e^{k_{\nu_m}vt\partial} H_m(k_{\nu_m}a) e^{-i\nu_m t} + \int B(\mu_m) e^{k_{\mu_m}vt\partial} H_m(k_{\mu_m}a) e^{-i\mu_m t} d\mu_m = 0\,. \tag{65}$$

Up to and including first-order terms in kvt vanish in (65): They do not appear in the expansion of the terms in (65) outside the integral, because this is the way these terms have been constructed—to mutually cancel $(kvt)^0$ and $(kvt)^1$ terms. Hence the terms in the integral of zero and first order in kvt, must be assigned a coefficient zero. Second- and third-order terms in kvt are retained in the expansion of (65), yielding

$$\frac{(kvt)^2}{2}\left[\partial^2 J_m(ka)\exp\left(-i\omega t + \frac{kvt}{3}\frac{\partial^3 J_m(ka)}{\partial^2 J_m(ka)}\right) - \frac{J_m(ka)}{H_m(k_\nu a)}\partial^2 H_m(k_{\nu_m}a) \times \exp\left(-i\nu_m t + \frac{kvt}{3}\frac{\partial^3 H_m(k_{\nu_m}a)}{\partial^2 H_m(k_{\nu_m}a)}\right)\right], \tag{66}$$

for the terms in (65) outside the integral. In the integral the expansion of (65) and keeping the relevant term yields

$$\frac{(vt)^2}{2}\int B(\mu_m)\, k_{\mu_m}^2 \partial^2 H_m(k_{\mu_m}a) \times \exp\left(-i\mu_m t + \frac{kvt}{3}\frac{\partial^3 H_m(ka)}{\partial^2 H_m(ka)}\right) d\mu_m\,. \tag{67}$$

By comprising (66) and (67) two new complex frequencies are prescribed, and two coefficients $B_1(\mu_{1m})$, $B_2(\mu_{2m})$. The details become cumbersome and will not be further pursued here. The solution (64) is now supplemented by adding the terms involving $B_1(\mu_{1m})$, $B_2(\mu_{2m})$. The process can be repeated to derive higher-order approximations, but the bookkeeping and the additional detail render it prohibitive within the scope of the present study. In principle, the creation of the new poles due to the new coefficients has been demonstrated. Thus the effect of the motion can be detected by extraction of poles from the impulse response.

V. THE SPHERE

In the context of electrodynamics, scattering by a sphere is an extremely complicated problem involving vector waves (see Stratton[23]). In acoustics, e.g., see Morse and Ingard,[24] we are dealing with a scalar problem not much different from the cylinder. The pertinent special functions are more complicated to handle, hence examples are confined to simple cases.

The general solution of the scalar wave equation for spherical systems and time harmonic fields is

$$u_\omega(\mathbf{r}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn} z_n(kr) Y_n^m(\theta,\phi)\, e^{-i\omega t},$$
$$Y_n^m(\theta,\phi) = P_n^m(\cos\theta)\, e^{im\phi},$$
$$P_n^{-m} = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m, \qquad (68)$$

where z_n denotes the spherical Bessel functions j_n, or spherical Hankel function of the first kind h_n, Y_n^m are scalar spherical harmonics, and P_n^m are the associated Legendre polynomials.

For the simple problem of the radially pulsating sphere, the three-dimensional analog of (38), the coordinates θ,ϕ are not affected, hence the incident wave (14)

$$u_i = \sum_{n=0}^{\infty} i^n(2n+1)\, P_n(\cos\theta)\, j_n(kr) e^{-i\omega t}, \qquad (69)$$

prescribes

$$u_s = \sum_n \int A_n(\nu) i^n(2n+1)\, P_n\, j_n(k_\nu r) e^{-i\nu t}. \qquad (70)$$

The argument proceeds along the same lines of the pulsating cylinder, and we end up with the analog of (44),

$$J_u(-ik\xi\partial)\, j_n(ka) + \sum_q A_{qn}\, J_{u-q}(-ik_q\xi\partial) h_m(k_q a) = 0, \qquad (71)$$

where k_q is given in (44).

The other problem which does not perturb the θ,ϕ structure of the solution is the radially expanding sphere moving according to (56). There is no need to retrace the argument, merely to notice that for the present case (62), (63), (66), and (67) apply with J_m, H_m replaced by j_n, h_n.

Other classes of problems can be defined, for which the object's surface is perturbed as a function of ϕ or θ. The apparently simple mode (50), corresponding here to azimuthal surface waves, follows the same argument (51)–(55), preserving the dependence on θ at each stage. The tedious details are not given here. Surface waves depending on the polar coordinate, e.g., (50) with $\sin\phi$ replaced by $\sin\theta$ or $\cos\theta$ are even more complicated, involving recurrence relations on the scalar spherical harmonics. Simple approximations are discussed by Censor.[11]

In all cases, it is evident that new coefficients appear, defining new poles and new complex frequencies in the impulse response of the perturbed sphere under consideration.

[1] T. P. Gill, *The Doppler Effect* (Academic, New York, 1965).
[2] D. Censor, "Doppler effect theory—fact, fiction and approximation," in *Proceedings of the URSI Symposium on Electromagnetic Theory*, Santiago de Compostela, Spain, 23–26 August 1983, pp. 17–24, to appear in Radio Sci. (1984).
[3] J. Van Bladel, *Relativity for Engineers* (Springer-Verlag, Berlin, 1984).
[4] D. Censor, "The generalized Doppler effect and applications," J. Franklin Inst. **295**, 103–116 (1973).
[5] J. Cooper, "Scattering of electromagnetic fields by a moving boundary: the one-dimensional case," IEEE Trans. Antennas Propag. **AP-28**, 791–795 (1980).
[6] J. Van Bladel and D. De Zutter, "Reflection from a linear vibrating metallic plate," IEEE Trans. Antennas Propag. **AP-29**, 629–636 (1981).
[7] D. Censor, "The Doppler effect for scattering by plane boundaries at normal incidence," and reply by J. Van Bladel and D. De Zutter, IEEE Trans. Antennas Propag. **AP-29**, 825 (1981).
[8] D. De Zutter, "Reflection from linearly vibrating objects—plane mirror at oblique incidence," IEEE Trans. Antennas Propag. **AP-30**, 898–904 (1982).
[9] D. Censor, "Velocity effects in scattering from expanding bubbles," J. Sound Vib. **21**, 379–385 (1972).
[10] D. Censor, "Scattering from expanding systems," Israel J. Tech. **11**, 109–115 (1973).
[11] D. Censor, "Scattering by time varying obstacles," J. Sound Vib. **25**, 101–110 (1972).
[12] P. H. Rogers, "Comments on Scattering by time varying obstacles," J. Sound Vib. **28**, 764–768 (1973), and author's reply.
[13] C. E. Baum, "The singularity expansion method," in *Transient Electromagnetic Fields*, edited by L. B. Felsen (Springer-Verlag, Berlin, 1976).
[14] C. E. Baum, "Emerging technology for transient and broadband analysis and synthesis of antennas and scatterers," Proc. IEEE **64**, 1598–1616 (1976).
[15] C. E. Baum, "Towards an engineering theory of electromagnetic scattering: The Singularity and eigenmode expansion methods," in *Electromagnetic Scattering*, edited by P. L. E. Uslenghi (Academic, New York, 1978), pp. 571–651.
[16] C. L. Dolph and R. A. Scott, "Recent developments in the use of complex singularities in electromagnetic theory and elastic wave propagation," in *Electromagnetic Scattering*, edited by P. L. E. Uslenghi (Academic, New York, 1978), pp. 503–570.
[17] C. L. Bennett and G. F. Ross, "Time-domain electromagnetics and its applications," Proc. IEEE **66**, 299–318 (1978).
[18] E. K. Miller and J. A. Landt, "Direct time-domain techniques for transient radiation and scattering from wires," Proc. IEEE **68**, 1396–1423 (1980).
[19] D. Censor, "Transient scattering in velocity dependent systems" (in process).
[20] V. Twersky, "Scattering of waves by two objects," in *Electromagnetic Waves*, edited by R. E. Langer (Univ. Wisconsin P., Madison, WI, 1962), pp. 361–389.
[21] V. Twersky, "Multiple scattering by arbitrary configurations in three dimensions," J. Math. Phys. **3**, 83–91 (1962).
[22] V. Twersky, "Multiple scattering of electromagnetic waves by arbitrary configurations," J. Math. Phys. **8**, 589–610 (1967).
[23] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[24] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968).