Generalized Doppler effect: Coherent and incoherent spectra

Dan Censor
Department of Electrical and Computer Engineering Ben Gurion University of the Negev, Beer Sheva, Israel 84105

Vernon L. Newhouse
Department of Electrical and Computer Engineering and Biomedical Engineering and Science Institute, Drexel University, Philadelphia, Pennsylvania 19104

(Received 2 September 1987; accepted for publication 11 January 1988)

The generalized Doppler effect spectrum produced by particles moving on arbitrary trajectories is analyzed. Scattering by a single particle moving in the farfield of a transmitter and a receiver is considered. It is shown that the Doppler spectrum exciting the particle is due to two factors: The longitudinal (or radial) component of the motion produces a spectrum similar to the conventional Doppler effect, by affecting the phase of the excitation wave. In addition, spectral effects are produced by the particle moving transversely (or angularly) through the spatially modulated radiation pattern of the transmitter. By virtue of the reciprocity properties of transmitters and receivers, each spectral component of the excitation signal again gives rise to spectra induced by the longitudinal and the transversal components of the motion relative to the receiver. The combined Doppler spectrum observed at the receiving transducer or antenna output is a convolution of all four spectra. The behavior of an ensemble of scattering particles is analyzed. The statistics are found by defining the particle and/or the trajectory parameters as random variables. Presently, it is assumed that the particles are identical, their positions are uncorrelated, and multiple scattering is ignored. It is shown that the interference of scattered waves from various particles, manifested in the coherent radiation, gives rise to a spectrum that might be different from that of a single particle. In special cases, the combined spectrum degenerates into a single frequency, and when this is the transmitter's frequency, the Doppler effect completely disappears. This explains the fact that when we have moving media, but the boundary surfaces are at rest, there is no Doppler effect. The results of the present analysis contribute to our understanding of Doppler velocimetry methods using ultrasound or laser radiation in medical and industrial instrumentation.

PACS numbers: 43.20.Hq, 43.28.Bj, 43.30.Es

INTRODUCTION

Previous studies of wave propagation in the presence of moving media tacitly or explicitly assumed that there are no Doppler frequency shifts when interfaces are at rest, although the media might be in motion. This approach neglected the fact that to some extent media always contain inhomogeneities, whose motion produces Doppler frequency shifts. Ultrasound or laser Doppler measurements are based on the latter phenomenon. This paradoxical situation is studied here from the point of view of the scattering produced by discrete particles, and the coherent and incoherent spectra created by an ensemble of such particles.

We deal here with the generalized Doppler effect produced by particles moving on arbitrary trajectories, having arbitrarily varying velocities, and illuminated by arbitrary sources. First, the generalized Doppler effect produced by a single particle is analyzed. It is shown that to the first order in v/c, the spectrum measured by the receiver is a convolution of the particle excitation spectrum and the spectrum produced by the particle as it moves relative to the receiver. Each of these spectra can themselves be represented as a convolution of two spectra, due to the longitudinal and transversal components of the motion relative to the transmitter or the receiver.

The statistical aspect is introduced by defining particle or trajectory parameters as random variables and specifying their associated probability density functions. By defining an ensemble of noninteracting particles, we create an equivalent scattering object whose shape, density, and motion are specified. The individual particles act as infinitesimal volume elements in the sense of a first-order Born approximation. It is then possible to discuss, for example, equivalent scatterers in the shape of rotating cylinders of circular and elliptical cross section, and derive the associated Doppler spectra.

It is shown below that in some cases the coherent Doppler spectrum degenerates into the incident field's frequency, i.e., the Doppler effect disappears. However, this cannot happen to the incoherently scattered field where intensities (as opposed to phase-dependent field magnitudes in the coherent case) are summed. Consequently, the experimenter using Doppler methods must always determine whether it is the coherent or the incoherent field that is predominant, or to what extent it is important to separate the coherent and incoherent components of the measured radiation.

The fact that the Doppler spectrum produced by scattering by a single particle and the coherent spectrum produced by a collection of statistically identical particles can be

 0001-4966/88/062012-08$00.80

quite different has been pointed out recently by Censor and Le Vine.[1] An example has been analyzed, involving an ensemble of particles moving inside a slab region parallel to the interfaces. It has been shown that although the field scattered by individual particles displays Doppler frequency shifts, the ensemble averaged field, i.e., the coherent radiation, contains the incident frequency only. In that special case, the disappearance of the Doppler effect was due to the fact that the coherent scattered field is reflected in the specular direction. Conditions have been discussed for which the Doppler effects will reappear, e.g., when the slab region is masked by an opaque screen with a small aperture. This constitutes only a special example because in many diffraction problems the Doppler effect will disappear even though no specular scattering occurs.

It is interesting to investigate these situations where the Doppler effect changes gradually, because many studies involving moving media do not even mention the possibility of Doppler effects. In other cases, when the lack of Doppler effects is stipulated, it is sometimes explained by the fact that the boundary surfaces are at rest, although the media are in motion. It has been observed[1] that this is true as a limiting case, at best. Many studies belonging to this category and involving electromagnetic waves are cited by Van Bladel.[2] Similar problems involving acoustical and elastic waves have been considered by Morse and Ingard[3] (see Chap. 11), Censor and Aboudi,[4] Steinmetz and Singh,[5] Stallworth and Jacobson,[6] Aboudi and Censor,[7] Censor *et al.*,[8] Censor and Schoenberg,[9] Schoenberg and Censor,[10,11] Savkar,[12] Dash,[13,14] Amiet,[15] Scott,[16] Koutsoyannis,[17] Koutsoyannis *et al.*,[18] Lyamshev,[19] and Candel.[20] Practically all above cited studies consider plane, circular, cylindrical, or spherical boundary surfaces, and because these surfaces are not perturbed by the tangentially moving media, it is understood that Doppler effects do not occur. It has been recently proposed[21] that Doppler effects produced by moving surfaces be investigated by assuming a collection of small, noninteracting particles distributed on such surfaces. This approach, amounting to a first-order Born approximation, is implemented here. First, a single particle moving along a deterministic specified trajectory is considered. The source illuminating this particle produces a field whose amplitude varies in space according to the radiation pattern associated with this source. As the particle moves through this space-modulated field, its excitation field varies with time. Therefore, even if the particle moves uniformly along a straight line, its excitation field will possess a broadened spectrum, not merely a single Doppler-shifted frequency. For the generalized Doppler effect this is compounded by the fact that for arbitrary trajectories and time-dependent velocities along them, a more complicated excitation signal is created, which yields a spectrum even when the radiation pattern is constant. In the farfield (i.e., in the sense of the Fraunhofer diffraction approximation), these effects enter in the phase and in the directivity function, i.e., the radiation pattern, of the excitation field. These effects constitute a product in the time domain, corresponding to a convolution of spectra in the frequency domain.

In order to determine the signal measured at the receiver, the reciprocity principle is invoked. For each spectral component of the excitation field, the particle now acts as a transmitter, and, as it moves along its trajectory, the signal at the receiver will be modulated, due to both the receiver's radiation pattern and the generalized Doppler effect arising from the nonuniform motion. Once more this yields a spectrum that is a convolution of two spectra. It is shown that to the first order in v/c, the received spectrum is a convolution of all four spectra.

The problem is very complicated, and for the electromagnetic case a relativistically exact treatment is out of the question at this time. As pointed out in the literature,[2,21] a quasispecial-relativistic treatment can be effected by considering the instantaneous velocity. However, this must be considered as heuristic and approximate, because to date no proof exists regarding the validity of this method. Furthermore, because of the nonuniform motion, the Lorentz transformation is completely abandoned and replaced by nonrelativistic considerations and the Galilean transformation. For electromagnetic waves, the relativistic amplitude effect and the time retardation can be incorporated into the present model in a heuristic manner. The strategy outlined above has sometimes been referred to[2,22,23] as solving the problem directly in the laboratory system of reference. Accordingly, we do not attempt to transform the excitation field into the particle's comoving frame of reference and express it in terms of the associated proper space and time coordinates $\mathbf{r}'$, t'. Instead, it is assumed that the scatterer, moving on the given trajectory $\mathbf{r}(t)$, is excited by the incident wave expressed in terms of $\mathbf{r}(t)$, t. Obviously this is exactly what would happen if a Galilean transformation and $t = t'$ were used. This treatment is legitimate for nonrelativistic mechanical waves, e.g., ultrasound. For electrodynamics, this approach is heuristic only. Unfortunately, it is not even clear to what extent it provides for an approximate solution. For lack of a better theory, it is used in many studies, for example, in analyzing rotating systems, see Van Bladel,[2] who also cites many relevant studies.

The rest of this article is devoted to discussion of a few simple examples. We are able to show, for example, that a rotating circular cylindrical shell produces no coherent Doppler frequency shifts. However, when the surface is perturbed such that a rotating elliptical cylinder is created, new sidebands appear in the scattered field. In order to demonstrate the gradual transition from a Doppler spectrum to its disappearance, the time delay between particles on a given trajectory is defined as a random variable with a Gaussian probability density function. This produces a Gaussian bandpass filter for the coherent radiation. By changing parameters, the width of this filter is varied, and in the extreme cases it can become an all pass or a δ-function notch filter.

I. GENERALIZED DOPPLER EFFECT FOR A SMALL OBJECT

Scattering by small particles in the acoustical field is discussed by Morse and Ingard.[3] The corresponding problem for the electromagnetic field is also amply discussed, e.g., see Stratton[24] and Jackson.[25] The electromagnetic problem is more complicated due to its vectorial nature. The essential features of the generalized Doppler effect will be retained if we confine the argument to scalar waves. The acoustical and electromagnetic problems can be presented simultaneously for the case of scattering by thin cylinders with the incident wave propagating perpendicularly to the axis. In the electromagnetic case, a scalar problem exists for normal incidence and E or H field polarization along the cylindrical axis. In any case, we shall consider the thin cylindrical scatterer to act as a monopole, i.e., an omnidirectional scatterer. The present approach also simplifies the analysis because the trajectory

$$\mathbf{r} = \mathbf{R}(t) \tag{1}$$

now lies in the xy plane, perpendicular to the axis z; hence, we have only two components

$$x = X(t)\,,\qquad y = Y(t)\,, \tag{2}$$

to consider. The source emits a monochromatic wave ψ_S at frequency ω_0, which at large distances can be written in the form

$$\psi_S = e^{-i\omega_0 t_S} g_S(\hat{\mathbf{r}}_S)/r_S^{1/2}\,, \tag{3}$$

where ψ stands for the field variable, e.g., the acoustic potential or the pressure in the sound field, or E_Z in the electromagnetic field; r_S is the distance from the source; and

$$t = t_S + r_S/c \tag{4}$$

is the retarded time relevant to the scatterer at distance r_S, with c as the phase velocity. In (3), g_S describes the scattering amplitude, which is a function of the excitation frequency and direction. For the present two-dimensional case, the scattering amplitude is as a function of ϕ, the azimuthal angle about the cylindrical axis. The trajectory (1), (2) discussed here is confined to a small region of space, such that its dimensions are small compared to r_S. Consequently, (3) can be considerably simplified, in a manner that will enable us to directly substitute (1) in (3). First, note that due to the restricted motion of the particle, r_S may be taken as a constant in the denominator of (3). Second, the phase

$$-\omega_0 t_S = k_0 r_S - \omega_0 t \approx \mathbf{k}_0 \cdot \mathbf{r}_S - \omega_0 t, \quad k_0 = \omega_0/c \tag{5}$$

can be written as a plane-wave phase. This (local) plane-wave front propagates in direction $\mathbf{k}_0$. The location $\mathbf{r}_S$ in (3) can now be identified with the trajectory defined in (1). This yields the phase of (3) at the moving particle. Assuming $\mathbf{r}_S = \mathbf{x}$, (3) can be rewritten as

$$\psi_S = e^{-\omega_0 t + ik_0 x} g_S(y/r_{S0})/r_{S0}^{1/2}\,, \tag{6}$$

where r_{S0} is a constant and ϕ is approximated by y/r_{S0}. The approximate representation (6) is adequate for localized motion in the farfield. Since the effect of a converging lens is to transform angles to distances in the focal plane, this representation is also adequate near the focus of a focused beam, with r_{S0} standing for the focal length. The source signal exciting the moving scatterer is obtained by substituting (2) in (6):

$$\psi = e^{-i\omega_0 t + ik_0 X(t)} g_S[Y(t)/r_{S0}]/r_{S0}^{1/2} \equiv e^{-i\omega_0 t}\int \Psi(\omega) e^{-i\omega t}\, d\omega = \int \Psi(\omega' - \omega_0) e^{-i\omega' t}\, d\omega'\,, \tag{7}$$

where $\omega = \omega' - \omega_0$ and Ψ is the Fourier transform of the signal ψ downshifted by an amount ω_0.

In the electromagnetic case, the motion also affects the amplitude of the wave.[2] Assuming the validity of the relativistic formalism for instantaneous velocities, to the first order in the velocity, a factor

$$1 - 2\frac{d}{dt}\frac{X(t)}{c}$$

should multiply g_S in (7). This effect was predicted for constant velocity by Einstein[26]; however, this has been already known before the advent of his theory (see, for example, Censor[21] for early references) and is heuristically extended to arbitrary instantaneous velocities.[2,22,23] The above factor can be incorporated into the present formalism at any time. However, to keep the model as simple as possible, for the time being, it will be omitted.

The excitation signal (7) involves two spectra defined by the Fourier transforms

$$e^{ik_0 X(t)} \equiv \int \xi(\omega) e^{-i\omega t}\, d\omega\,,\qquad g_S\frac{[Y(t)/r_{S0}]}{r_{S0}^{1/2}} \equiv \int \eta(\omega) e^{-i\omega t}\, d\omega\,. \tag{8}$$

In view of (7), the spectrum Ψ is given by the convolution

$$\Psi(\omega) = \xi(\omega) * \eta(\omega)\,. \tag{9}$$

The factor $e^{-i\omega_0 t}$ shifts the frequencies to the reference ω_0.

In order to compute the scattered field at the receiver, the reciprocity principle is invoked. Each scattered spectral component ω' is now considered to be radiated by a transmitter. In analogy to (3), we now have a signal

$$\Psi(\omega' - \omega_0) e^{-i\omega' t} g(\omega')/r^{1/2}, \quad \omega' = \omega_0 + \omega\,, \tag{10}$$

where $g(\omega')$ is the scattering amplitude of the moving object. Inasmuch as we chose monopoles, the argument is reserved to display the dependence on the frequency; t is the time referred to the scatterer and r is the distance from the scatterer. The scatterer amplitude of a monopole is independent of direction but depends on frequency, and thus affects the shape of the spectrum. To investigate this phenomenon, consider the linear approximation

$$g(\omega') = g(\omega_0) + \frac{\partial g}{\partial \omega_0}(\omega' - \omega_0) = g(\omega_0)\left(1 + \frac{\omega\, \partial g/\partial\omega_0}{g(\omega_0)}\right), \tag{11}$$

obtained by expanding $g(\omega')$ in the vicinity of ω_0 and retaining the first derivative only. This should be compared to

$$\Psi(\omega) = \Psi(0)\left(1 + \frac{\omega(\partial\Psi/\partial\omega)_{\omega=0}}{\Psi(0)}\right), \tag{12}$$

which appears as a factor in the same expression (10). Note that both (11) and (12) are expanded about the reference frequency ω_0. For small particles $g(\omega')$ is proportional to $(\omega')^n$, where n is some power, e.g., for Rayleigh scattering $n = 2$, hence, the derivative of $g(\omega')$ is proportional to $2\omega'$. It follows that the term in parentheses in (11) is $1 + 2\omega/\omega_0$. For a narrow spectrum, this expression will be approximately 1: In other words, in (10), we need not take into account the dependence of the scattering amplitude on frequency. On the other hand, $\Psi(\omega) = \Psi(\omega' - \omega_0)$ in (10) is expanded in (12) about $\omega = 0$. Suppose we have the same structure of $\Psi(\omega)$ proportional to ω^2, then

$$\omega \frac{(\partial\Psi/\partial\omega)}{\Psi(\omega)} = 2$$

and is not negligible compared to 1. In general, because $\Psi(\omega)$ is centered about $\omega = 0$, it is not a narrow-band spectrum, and the approximation that works for (11) is inadequate for (12). The signal measured at the receiver at $r = r_R$ depends on the retarded time t_R,

$$t_R = t + r_R/c , \quad (13)$$

and the receiver's radiation pattern

$$g_R(\hat{\mathbf{r}}_R) ,$$

which, similar to (11), is assumed to be frequency independent for narrow-band Doppler spectra. As in (5), we approximate

$$\omega t_R = kr_R + \omega t \approx \mathbf{k}_R \cdot \mathbf{r}_R + \omega t, \quad \mathbf{k}_R = \hat{\mathbf{r}}_R \omega/c = \hat{\mathbf{r}}_R k , \quad (14)$$

and identify $\hat{\mathbf{r}}_R = \hat{\mathbf{x}}'$, and $\hat{\mathbf{y}}'$ such that $\hat{\mathbf{x}}' \times \hat{\mathbf{y}}' = \hat{\mathbf{z}}'$. Corresponding to (2), we now have

$$x' = X'(t_R), \quad y' = Y'(t_R) . \quad (15)$$

Note that $\mathbf{r}_S$ is oriented from the source of the scatterer, while $\mathbf{r}_R$ points from the scatterer towards the receiver. For the case of coinciding transmitter and receiver, we have $\mathbf{r}_S = -\mathbf{r}_R$ and $X(t)$, $Y(t)$ and $X'(t_R)$, $Y'(t_R)$ describe mirror image trajectories, with t_R occurring at a later time, according to (13). This means that for an observer attached to the scatterer's frame of reference, the receiver now appears to be moving.

In analogy to (7), the received signal due to a single frequency ω' is given by

$$(g/r_R^{1/2})\Psi(\omega)e^{-i\omega' t_R + ik'X'(t_R)} g_R[Y'(t_R)/r_R] . \quad (16)$$

Similarly to (8) we now define

$$e^{ik'X'(t_R)} \equiv \int \xi_R(\nu)e^{-i\nu t_R}\, d\nu ,$$
$$\frac{g}{r_{R0}^{1/2}} g_R\left(\frac{Y'(t_R)}{r_R}\right) \equiv \int \eta_R(\nu)e^{-i\nu t_R}\, d\nu . \quad (17)$$

Note carefully that both expressions in (17) depend on ω', the first because it contains k' and the second because g_R also depends on the excitation frequency ω'. However, since $X'(t_R)$, $Y'(t_R)$ are already of first order in the velocity, we are justified in approximating in these expressions $\omega' = \omega_0$. We therefore derive the receiver signal $\psi_R(t_R)$ in the form

$$\psi_R = \left(\frac{g}{r_{R0}^{1/2}} \int \xi_R(\nu)e^{-i\nu t_R}\, d\nu\right)\left(\int \eta_R(\nu)e^{-i\nu t_R}\, d\nu\right) \times \left(\int \Psi(\omega)e^{-i\omega t_R}\, d\omega\right)e^{-i\omega_0 t_R} . \quad (18)$$

From (9), (18) it is now clear that the spectrum corresponding to $\Psi_R(t_R)$ is given by

$$\Psi_R(\omega) = [g/(r_{S0}r_{R0})^{1/2}]\xi(\omega)*\eta(\omega)*\xi_R(\omega) *\eta_R(\omega)*\delta(\omega - \omega_0) . \quad (19)$$

This is a very interesting and analytically convenient result, describing the total spectrum as the convolution of the individual spectra. Special simple cases of (19) have been noted before; see Newhouse *et al.*[27] and Bascom *et al.*[28] Inasmuch as a single object is considered, we are dealing here with coherent radiation. The incoherent radiation appears when an ensemble is introduced, as will be done below.

II. TWO SIMPLE EXAMPLES

In order to highlight the result (19), two simple examples will be considered. The first involves simple uniform motion and effects on the spectrum that are introduced by the transmitter and receiver radiation patterns. The second example involves omnidirectional transmitter and receiver, so that in this case the radiation patterns have no effect, however, the nonuniform motion itself will have an effect on the spectrum. For this example a harmonic motion is chosen.

Example 1: Let the transmitter and receiver be situated at the same location (actually, for pulsed radar and ultrasound pulsed Doppler systems the same element can be used both for transmitting and receiving, on a time sharing basis). Let the transmitter and receiver apertures be infinite strips defined by $-W/2 \leqslant y \leqslant W/2$ at a distance $r = x_0$ from the scatterer. The transmitter's farfield radiation pattern, i.e., in the sense of the Fraunhofer diffraction approximation, is given by the Fourier transform of the aperture function (see, for example, Born and Wolf[29]) and the farfield is, therefore,

$$x_0^{-1/2}e^{ik_0x_0 - i\omega_0 t}\,\mathrm{sinc}[k_0(W/2)\sin\varphi], \quad \sin\varphi \approx y/x_0 . \quad (20)$$

For a scatterer moving according to $y = vt$, i.e., performing transverse motion across the beam, in the focal plane, at a distance x_0 from the transmitter, the excitation field is proportional to

$$x_0^{-1/2}e^{ik_0x_0 - i\omega_0 t}\,\mathrm{sinc}\left(\omega_0 t\, \frac{W}{2x_0}\frac{v}{c}\right), \quad (21)$$

corresponding to a boxcar function spectrum, constant in the range

$$\omega_0\left(1 - \frac{W}{2x_0}\frac{v}{c}\right) \leqslant \omega \leqslant \omega_0\left(1 + \frac{W}{2x_0}\frac{v}{c}\right) . \quad (22)$$

Since the motion is only in the y direction and the transmitter and receiver coincide on the x axis, we have

$$\xi(\omega) = \xi_R(\omega) \quad (23)$$

proportional to $\delta(\omega - \omega_0)$, and $\eta(\omega) = \eta_R(\omega)$ are boxcar functions over the range of (22). Now, according to (19), the convolution of the rectangular spectra $\eta(\omega)$, $\eta_R(\omega)$, (22) yields an equilateral triangular spectrum profile in the range

$$\omega_0\left(1 - \frac{W}{x_0}\frac{v}{c}\right) \leqslant \omega \leqslant \omega_0\left(1 + \frac{W}{x_0}\frac{v}{c}\right) \quad (24)$$

with the peak at ω_0. The sample was analyzed by Newhouse *et al.*[27] and demonstrates the theory for the case of a variable field and constant velocity.

Example 2: Here the transmitter and receiver radiation patterns have no effect on the spectrum. In practice, this implies a broad mainlobe in the radiation pattern. In the mathematical model, we take g_S, g_R as constants; hence, according to (8) and (17)

$$\eta(\omega) = \eta_R(\omega) = \delta(\omega - \omega_0) . \quad (25)$$

The transmitter–receiver pair are situated as in the first example. The motion is described by

$$x = X(t) = x_0 + a \cos \Omega t , \quad (26)$$

where

$$a \ll x_0$$

is a constant. Following from (8), we have

$$e^{ik_0 a \cos \Omega t} = \sum_{n=-\infty}^{\infty} i^m J_m(k_0 a) e^{im\Omega t} ,$$

$$\xi(\omega) = \sum_{n=-\infty}^{\infty} i^m J_m(k_0 a) \delta(\omega - m\Omega) , \quad (27)$$

where J_m are the nonsingular Bessel functions. Due to $\delta(\omega - \omega_0)$ in (19), we have $\delta(\omega_0 - m\Omega)$ in (27), which means that the Doppler effect produces discrete sidebands at intervals Ω on the ω axis. Except for small a, (27) cannot be considered as a narrow-band spectrum as stipulated above. For small arguments

$$J_m(k_0 a) = (-1)^m J_{-m}(k_0 a) \approx \frac{1}{m!}\left(\frac{k_0 a}{2}\right)^m , \quad m = 0,1,2,..., \quad (28)$$

is used, which can be truncated, thus describing a bandlimited spectrum. According to (15) and (17), we now have $x' = -x$, but the directions of propagation are opposite; hence, we get (27) once more, except for the time, which is now t_R. In the time domain, the received signal is proportional to

$$e^{ik_0 a(\cos \Omega t + \cos \Omega t_R)} , \quad (29)$$

but, for small a, we have

$$\cos \Omega t_R \approx \cos \Omega(t + x_0/c) ,$$

whereby we have included in the parentheses of (29) only the zero-order expansion of t_R, (26), because the exponent already contains a as a factor. Accordingly, (29) becomes

$$\sum_m \sum_{m'} i^{m+m'} J_m(k_0 a) J_{m'}(k_0 a) e^{i(m+m')\Omega t + im' x_0/c} \quad (30)$$

displaying the same discrete sideband structure. Problems of this kind have been considered before in connection with vibrating objects; see references in Van Bladel,[2] and Censor.[21,22,30] Note that in this example, only $X(t)$, i.e., the projection on the line of sight, plays a role. This means that $Y(t)$, the transverse motion, may be arbitrary. In particular, one would obtain the same effect for a rotating object whose motion is prescribed by (26) and

$$y = Y(t) = b \sin \Omega t , \quad (31)$$

where

$$b \neq a$$

describes elliptical trajectories, and, in the limit $b = a$, (26) and (31) describe a circle. This terminates the discussion for the coherent radiation produced by a single scatterer. The many particle ensemble is introduced in the next section.

III. INCOHERENT AND COHERENT SCATTERING

The statistical aspects of scattering are amply discussed in the literature. For an extensive list of earlier references see Ishimaru.[31] Inasmuch as we are mainly interested in the Doppler effects, all other facets of scattering by random ensembles will be kept as simple as possible. The scatterers are assumed to be identical in all their properties, except for their locations, which are assumed to be random and uncorrelated. This means that the scatterers are small enough, such that the location of one does not affect the probable position of another. If this assumption is invalid, e.g., for closely packed finite size particles, then pair correlation (and eventually higher-order correlations) considerations must be included. Also, multiple scattering (see Twersky[32] and also review of Twersky's work in Ishimaru[31]) is ignored here, assuming a sparse ensemble with large interparticle distances.

For incoherent scattering, the effect of the ensemble of uncorrelated scatterers on the spectrum is discussed in terms of the power contributed by individual scatterers. Following Twersky[31] we define a weighting function, or probability density function, $p(V)$ describing the probability of various parameters in representation space V associated with the scatterer and the trajectory, e.g., the particle density at location $\mathbf{r}$. We then average $|\Psi|^2 = \Psi_R \Psi_R^*$ over the representation space V of all random variables, obtaining

$$S_R = \int_V \Psi_R(\omega) \Psi_R^*(\omega) p \, dV , \quad (32)$$

and define $S_R(\omega)$ as the (total) power spectral density. The corresponding amplitude spectral density is the square root $[S_R(\omega)]^{1/2}$. It is of course well known (for instance, Papoulis[33]) that, if we deal with wide sense stationary processes, $S_R(\omega)$ and the autocorrelation function form a Fourier transform pair. These are the well-known Wiener–Khinchin relations. It is also known (Papoulis,[33] p. 275 ff.) that the average power of the signal in the vicinity of some frequency ω_1 is approximately equal to $S_R(\omega_1)$, which is tantamount to stating that in deriving the power spectral density the phase is not taken into account.

On the other hand, the coherent power spectral density is derived by ensemble averaging the amplitude spectrum

$$\overline{\Psi}_R(\omega) = \int_V \Psi_R(\omega) p \, dV \quad (33)$$

and squaring the result. Since $\Psi_R(\omega)$ is the (complex) Fourier transform of the amplitude of the received signal at frequency ω, the summation (33) constitutes a phasor addition and allows for interference of waves emanating from different scatterers having various positions and velocities. The coherent spectrum (33) may even totally disappear while the remaining part of the total power spectrum, i.e., the incoherent power spectrum, does not vanish. It is the aim of the present section to investigate and compare the coherent and total power spectral densities under various circumstances.

An interesting example is provided by randomizing the initial time of motion. Thus instead of (1) we now have

$$\mathbf{r} = \mathbf{R}(t - \tau) , \quad (34)$$

where τ is a random variable associated with a probability density function $p(\tau)$. The effect on (19) subject to (8), (17) is to introduce a delay factor $e^{i\omega\tau}$, yielding

$$\Psi_{R,\tau} = [g/(r_{S0}r_{R0})^{1/2}]\xi(\omega)e^{i\omega\tau}*\eta(\omega)e^{i\omega\tau}*\xi_R(\omega)e^{i\omega\tau} *\eta_R(\omega)e^{i\omega\tau} = \Psi_R(\omega)e^{i\omega\tau}, \quad (35)$$

which is easily verified by using the definition of the convolution integral. Due to the complex conjugation the delay time has no effect on the incoherent spectrum (32), but affects the coherent spectrum (33). Consider the case of a Gaussian distribution for τ,

$$p(\tau) = (2\pi\sigma^2)^{-1/2}e^{-\tau^2/2\sigma^2}, \quad (36)$$

where σ is the standard deviation. From (33), (35), and (36), we obtain

$$\overline{\Psi}_{R,\tau}(\omega) = \Psi_R(\omega)e^{-\sigma^2\omega^2/2}. \quad (37)$$

Hence, the coherent portion of the total Doppler spectrum is affected by a "bandpass filter" centered about $\omega = 0$, i.e., $\omega' = \omega_0$. In the limit where $p(\tau)$ is taken as $\delta(\tau)$, we obtain $\Psi_R(\omega)$. On the other hand, for

$$\sigma \rightarrow \infty,$$

we obtain $p(\tau)$ increasingly flatter. Finally, if we take $p(\tau) = \text{const}$, (33) yields

$$\overline{\Psi}_{R,\tau}(\omega) = \Psi_R(\omega)\int_{-\infty}^{\infty} e^{i\omega\tau}\,d\tau = 2\pi\Psi_R(\omega)\delta(\omega). \quad (38)$$

This is a clear example showing how the coherent part of the Doppler spectrum reduces at the expense of the incoherent portion, due to the concerted effect of the ensemble particles. Previously, such an effect was rigorously computed only for the case of particles uniformly moving in a slab region.[1] More generally, if the particles move along a trajectory with constant speed, the situation may be construed as having a surface moving tangentially with respect to itself. This situation corresponds to the geometries of many of the articles cited above, where plane and cylindrical scatterers move parallel to their boundary, and circular cylinders and spheres rotate about the polar axis. The nonexistence of Doppler frequency shifts for such cases is usually explained by the fact that even though the media are in motion, the boundary surfaces are at rest. But this explanation does not always work. It has been argued[1] that if the surface is overlaid by a screen having a small aperture, the Doppler effects will reappear. In general, it would be more appropriate to say that all particles or parts of the medium produce individual Doppler spectra, but due to interference all frequencies except ω_0 are annihilated.

An important class of problems involves periodic trajectories, so that

$$\mathbf{R}(t) = \mathbf{R}(t - 2\pi n/\Omega), \quad (39)$$

where n is an integer and Ω is the (angular) frequency. Thus $\mathbf{R}(t)$ can be represented as a Fourier series

$$\mathbf{R}(t) = \sum \mathbf{A}_n \cos(n\Omega t) + \mathbf{B}_n \sin(n\Omega t) \quad (40)$$

and for $X(t)$, $Y(t)$ we have corresponding A_{nx}, B_{nx} and A_{ny}, B_{ny} coefficients. The exponentials in the first equation of (8) and (17) can now be represented as products of Bessel–Fourier series, e.g.,

$$\exp\left(ik_0 \sum_n A_{nx} \cos n\Omega t\right) = \prod_n \sum_m i^m J_m(k_0 A_{nx})e^{imn\Omega t}, \quad (41)$$

which once again displays the discrete spectrum with sidebands separated by Ω. If the initial time is randomized and taken as equiprobable between $-T/2$ to $T/2$, where T is the period, and (41) integrated between these limits according to (33), then the integral vanishes for all m, n, except $mn = 0$. Consequently, we have once again the situation where mutual interference annihilates the Doppler effects. For arbitrary periodic $Y(t)$, $Y'(t_R)$ in the second equation of (8), (17), respectively, periodic functions are obtained once more. Therefore, the argument following (41) applies here as well.

These remarks hold for continuous single or multiple frequency transmission. A special case results when the pulse and the particle moving on a closed trajectory have the same periodicity. It is easily visualized that in this case we get some kind of a "stroboscope" effect, in which pulses identical to previous ones, encounter the particles always in the same configuration in space, having the same velocities. This will produce a coherent spectrum. As in the case of a stroboscope, a coherent spectrum will be produced every time the incident pulses are synchronous with the trajectory repetiton frequency or a harmonic thereof. Also, the analogy is complete to the extent that if the pulse frequency is a subharmonic of the trajectory repetition frequency, we still get the coherent scattering, on each event where a pulse arrives and encounters the particles in the same configuration as for the previous pulse (note that in this case of the pulse frequency being a submultiple of the trajectory period, there are a few trajectory traverses which do not rise to scattered waves). Of course, this argument holds only for pulse repetition frequencies below the carrier frequency; otherwise the pulse repetition frequency loses its meaning and all kinds of inconsistencies arise.

Scattering problems involving rotating objects are of special interest, both for acoustical and electromagnetic problems. Some studies have been cited above.[2,4,9–11] See also De Zutter,[34,35] De Zutter and Goethals,[36] Goto and Shiozawa,[37–39] Petrov,[40] Rao,[41] Seikai and Shiozawa,[42] Tai,[43] and Van Bladel.[44,45] As an example, scattering by rotating cylinders will be considered. In order to be more concrete, a rotating elliptical cylinder will be considered. The ellipse is parametrically defined by means of

$$u = \alpha \cos\beta, \quad w = \epsilon\alpha \sin\beta, \quad (42)$$

where u, w are two Cartesian coordinates, α determines the size of the ellipse, and ϵ its eccentricity. The parameters α, β locate different points on the elliptical cylinder. A rotation transformation $x = u\cos\phi + w\sin\phi$, $y = -u\sin\phi + w \times \cos\phi$, and $\phi = \Omega t$ now define the rotating cylinder through

$$x = \alpha\cos\beta\cos\Omega t + \epsilon\alpha\sin\beta\sin\Omega t,$$
$$y = -\alpha\cos\beta\sin\Omega t + \epsilon\alpha\sin\beta\cos\Omega t. \quad (43)$$

For simplicity, let α be a constant, defining an elliptical shell.

The case of interest of a rotating circle is obtained by taking $\epsilon = 1$. Accordingly, (43) becomes

$$x = \alpha \cos(\Omega t - \beta), \quad y = -\alpha \sin(\Omega t - \beta). \qquad (44)$$

Evidently, β / Ω can be interpreted as τ, the initial time delay as in (34). It follows that if τ is evenly distributed in a 0–T interval, i.e., the circular shell is homogeneous, then forms similar to (41) are obtained and no Doppler frequency shifts will be created. The incoherent radiation will of course contain sidebands $\omega_0 \pm n\Omega$, $n = 1,2,...$. However, if the density $p(\beta)$ is not a constant, then in general the integral

$$\int_{-\pi}^{\pi} e^{in\beta} p(\beta) d\beta \qquad (45)$$

does not vanish except for $n = 0$ and all sidebands can be expected in the coherent spectrum. This corresponds to a case of a rotating circular cylinder with a varying surface impedance; see Petrov[40] and Van Bladel.[2] Furthermore, we can model an inhomogeneous circular cylinder by integrating over a range of α (i.e., in effect taking into account many concentric shells). For each shell, we will have $p(\alpha,\beta)$, i.e., an angular density distribution depending on β for the shell defined by some value of α. If the inhomogeneity is radially symmetrical, i.e., $p[\alpha,\beta(\alpha)]$ depends on α only, then although the shells may differ in density, each has a constant value for β, hence, Doppler frequency shifts will vanish once more. On the other hand, if the cylinder is eccentrically inhomogeneous, e.g., possesses an eccentric hole, then $\beta = \beta(\phi)$ depends on angles and all the sidebands $\omega_0 + - n\Omega$, $n = 1,2,...$, are expected in the received coherent signal.

Finally, consider the case of an ellipse with ε different from 1. Clearly the transition from (43) to (44) is now impossible. We are still able to recast expressions in Fourier–Bessel series, in a manner similar to (41). However, in this case β will appear in the arguments of the Bessel functions; hence, averaging will modify the amplitudes of the sidebands of the discrete spectrum, but in general the Doppler frequency shifts will not vanish.

IV. SUMMARY AND DISCUSSION

This study examined phenomena associated with Doppler effects produced by ensembles of small, uncorrelated and noninteracting (no multiple scattering) particles. In order to highlight the pertinent phenomena, the model is simplified from the statistical as well as from the field analysis points of view. Thus the transmitter and receiver are considered to be in the farfield. The model is also kept simple from the point of view of the physics involved. Questions regarding the proper transformations of coordinates and fields in the electromagnetic case are left open. Adequate modification of the present simple model can be performed if necessary; however, the rationale here is to highlight the Doppler effect aspects and keep everything else as simple as possible.

One of the main points discussed here is the nature of the Doppler spectrum produced by a particle moving on an arbitrary trajectory, at the same time traversing the radiation patterns of the transmitter and the receiver. To first order in v/c, the spectrum corresponding to this generalized Doppler effect is obtained by convolving the individual spectra involved. The second aspect discussed here is the effect of interference of the fields scattered by individual particles on the overall coherent spectrum. While the total power spectral density is determined by the power spectra of the individual particles, the coherent spectrum depends on the phase information as well. In various situations, the coherent Doppler spectrum degenerates into a single frequency, e.g., the original transmitter frequency. In order to analyze such configurations, "media" and "objects" are defined by randomizing trajectory parameters. For example, it is shown that for any trajectory, if the initial time is randomized, the coherent Doppler spectrum can be made to vanish, except for the transmitter's original frequency.

Special cases were discussed, the question of rotating cylinders, for example. It is shown that for homogeneous circular cylinders, rotating about their axis, the Doppler effect vanishes. However, if the cylinder is not homogeneous over its cross section, e.g., if the surface impedance depends on the azimuthal angle, or if an eccentric inhomogeneity is included, the Doppler effect is manifested by a discrete spectrum with the distance between sidebands corresponding to the rotation frequency. The elliptical cylinder, or in general any arbitrary cross section, except the circular cylinder rotating about the cylindrical axis, belongs to this category and produces the discrete spectrum. It is also shown that if the trajectory is periodic and synchronous with the transmitted radiation, the Doppler spectrum is completely coherent.

The present approach of discussing collections of small scatterers in order to gain insight into more complicated situations involving moving systems proves itself here as a very promising approach. Although multiple scattering is ignored, we thus are able to create "objects" and discuss the ensuing scattering spectra. Of course, in a more realistic analysis, one will have to discuss the effects introduced by the cloud of particles becoming increasingly dense.

ACKNOWLEDGMENT

This work was supported by NSF, Grant ECS-8219736, and by a Louis and Bessie Stein Family Foundation Fellowship for the first author.

[1]D. Censor and D. M. Le Vine, "The Doppler effect: Now you see it, now you don't," J. Math. Phys. **25**, 309–316 (1984).
[2]J. Van Bladel, *Relativity and Engineering* (Springer, Berlin, 1984).
[3]P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw-Hill, New York, 1968).
[4]D. Censor and J. Aboudi, "Scattering of sound waves by rotating cylinders and spheres," J. Sound Vib. **19**, 437–444 (1971).
[5]G. G. Steinmetz and J. J. Singh, "Reflection and transmission of acoustical waves from a layer with space-dependent velocity," J. Acoust. Soc. Am. **51**, 218–222 (1972).
[6]L. A. Stallworth and M. J. Jacobson, "Sound transmission in an isospeed ocean channel with depth-dependent current," J. Acoust. Soc. Am. **51**, 1738–1750 (1972).
[7]J. Aboudi and D. Censor, "Scattering of elastic waves by moving objects," J. Acoust. Soc. Am. **52**, 203–209 (1972).
[8]D. Censor, J. Aboudi, and D. Neulander, "Relection and transmission of elastic waves by a moving slab," Appl. Sci. Res. **25**, 313–326 (1972).
[9]D. Censor and M. Schoenberg, "Two dimensional wave problems in rotating elastic media," Appl. Sci. Res. **27**, 401–414 (1973).

[10]M. Schoenberg and D. Censor, "Velocity-dependent reflection, refraction, and scattering of elastic shear waves in the presence of a lubricating layer," J. Acoust. Soc. Am. **53**, 508–513 (1973).
[11]M. Schoenberg and D. Censor, "Elastic waves in rotating media," Appl. Math. **31**, 115–125 (1973).
[12]S. D. Savkar, "Radiation of cylindrical duct acoustic modes with flow mismatch," J. Sound Vib. **42**, 363–386 (1975).
[13]R. Dash, "Point source radiation through a composite medium," J. Sound Vib. **49**, 365–377 (1976).
[14]R. Dash, "Reflection and radiation due to a quadrupole near a fluid interface," J. Acoust. Soc. Am. **62**, 835–846 (1977).
[15]R. K. Amiet, "Refraction of sound by a shear layer," J. Sound Vib. **58**, 476–482 (1978).
[16]J. N. Scott, "Propagation of sound waves through a linear shear layer," Am. Inst. Aeronaut. Astronaut. J. **17**, 237–244 (1979).
[17]S. P. Koutsoyannis, "Characterization of acoustic disturbances in linearly sheared flows," J. Sound Vib. **68**, 187–202 (1980).
[18]S. P. Koutsoyannis, K. Karamcheti, and D. C. Galant, "Acoustic resonances and sound scattering by a shear layer," Am. Inst. Aeronaut. Astronaut. J. **18**, 1446–1454 (1980).
[19]L. M. Lyamshev, "Theory of sound propagation in a moving layered inhomogeneous medium," Sov. Phys. Acoust. **28**, 217–221 (1982).
[20]S. M. Candel, "Sound source radiation in two-dimensional shear flow," Am. Inst. Aeronaut. Astronaut. J. **21**, 221–227 (1983).
[21]D. Censor, "Theory of the Doppler effect: Fact, fiction and approximation," Radio Sci. **19**, 1027–1040 (1984).
[22]D. Censor, "The generalized Doppler effect and applications," J. Franklin Inst. **295**, 103–116 (1973).
[23]J. Van Bladel and D. De Zutter, "Reflection from a linear vibrating metallic plate," IEEE Trans. Antennas Propag. **AP-29**, 629–636 (1981).
[24]J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[25]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).
[26]A. Einstein, "Zur Elektrodynamik bewegter Koerper," Ann. Phys. (Leipzig) **17**, 891–921 (1905).
[27]V. L. Newhouse, D. Censor, T. Vontz, J. A. Cisneros, and B. Goldberg, "Ultrasound Doppler probing of flows transverse with respect to beam axis," IEEE Trans. Biomed. Eng. **BME-34**, 779–789 (1987).
[28]P. A. J. Bascom, R. A. C. Cobbold, and B. H. M. Roelofs, "Influence of spectral broadening on continuous wave Doppler ultrasound spectra: A geometric approach," Ultrasound Med. Biol. **12**, 387–395 (1986).
[29]M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1983).
[30]D. Censor, "Harmonic and transient scattering from time varying obstacles," J. Acoust. Soc. Am. **76**, 1527–1534 (1984).
[31]A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).
[32]V. Twersky, "On propagation in random media of discrete scatterers," in *Proceedings of the Symposium on Applied Mathematics* (Am. Math. Soc., Providence, RI, 1964), Vol. 16, pp. 86–116.
[33]A. Papoulis, *Probability, Random Variables and Stochastic Processes* (McGraw-Hill, New York, 1984).
[34]D. De Zutter, "Scattering by a rotating dielectric sphere," IEEE Trans. Antennas. Propag. **AP-28**, 643–651 (1980).
[35]D. De Zutter, "Scattering by a rotating circular cylinder with finite conductivity," IEEE Trans. Antennas Propag. **AP-31**, 166–169 (1983).
[36]D. De Zutter and D. Goethals, "Scattering by a rotating conducting sphere," IEEE Trans. Antennas Propag. **AP-32**, 95–98 (1984).
[37]H. Goto and T. Shiozawa, "Propagation of electromagnetic waves along a rotating dielectric column," Electron. Commun. Jpn. (English translation) **59-B**, 82–90 (1976).
[38]H. Goto and T. Shiozawa, "Relativistic solution to the problem of scattering by a rotating dielectric column," J. Appl. Phys. **49**, 556–561 (1978).
[39]H. Goto and T. Shiozawa, "Propagation of space charge waves along a rotating relativistic electron beam," Electron. Commun. Jpn. (English translation) **62-B**, 60–68 (1979).
[40]B. M. Petrov, "Spectral characteristics of the scatter field from a rotating impedance cylinder in uniform motion," Radio Eng. Electron. Phys. (USSR) (English translation) **17**, 1431–1437 (1972).
[41]T. C. K. Rao, "Electromagnetic wave propagation over a rotating dielectric cylinder," Can. J. Phys. (1986).
[42]S. Seikai and T. Shiozawa, "Scattering of electromagnetic waves by a rotating electron-plasma column," IEEE Trans. Antennas Propag. **AP-23**, 75–83 (1975).
[43]C. T. Tai, "Two scattering problems involving moving media," Ohio State University Res. Found., Rep. 1691–7 (1964).
[44]J. Van Bladel, "Electromagnetic fields in the presence of rotating bodies," Proc. IEEE **64**, 301–318 (1976).
[45]J. Van Bladel, "Rotating dielectric sphere in a low-frequency field," Proc. IEEE **67**, 1654–1655 (1979).