

# Velocity-dependent reflection, refraction, and scattering of elastic shear waves in the presence of a lubricating layer

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Wave-scattering problems involving moving media should take into account the effect of the boundary layers at the sliding interfaces. A recent paper by Graham and Graham [*J. Acoust. Soc. Amer.* 46, 169-175 (1969)], although restricted to acoustical waves, is indicative of the difficulties involved. Presently we consider shear waves in moving elastic media, having a viscous (Newtonian) fluid layer separating the sliding interfaces. The mean velocity profile in the viscous medium is prescribed by the solution of the relevant Navier-Stokes equation and is unaffected by the perturbations. For the mean velocity profiles encountered between parallel planes and between concentric cylinders, solutions for shear waves are obtained in terms of Bessel functions. Various limiting cases are investigated.

SUBJECT CLASSIFICATION: 11.2, 11.7, 11.8.

## INTRODUCTION

It is now accepted that the appropriate boundary conditions for acoustic waves at the boundary between moving media should be taken according to the theory of Miles<sup>1</sup> and Ribner.<sup>2</sup> Their ideas have been applied by Yeh.<sup>3,4</sup> While it was clear from the start that the acoustical pressures should be equated at the boundary, some earlier investigators failed to apply correctly the condition of continuity at the boundary. Friedland and Pierce<sup>5</sup> realized that the latter is tantamount to the requirement that the component of the particle displacement (rather than the velocity) must be continuous across the boundary. This has been adapted by Censor and Aboudi,<sup>6</sup> who treat the problem of scattering by rotating cylinders and spheres. The expectation that the problem of sliding elastic media will in the limit go over to the acoustical case prompted Aboudi and Censor,<sup>7</sup> and Censor, Aboudi, and Neulander<sup>8</sup> to assume that at the boundary, tractions, and displacements are continuous on the undeformed boundary. On the other hand, Samuels,<sup>9</sup> being aware of the work by Miles<sup>1</sup> and Ribner,<sup>2</sup> assumes different boundary conditions between moving elastic media for the plane strain case.

In the acoustical case, Graham and Graham<sup>10</sup> take a different approach, considering the propagation of acoustic waves in a shear layer in which the velocity changes linearly. This introduces the problem of wave propagation in nonuniformly moving media. However, the problem treated is purely acoustical, and the shear properties of the medium are neglected. In spite of this simplification, no closed-form solution is available. This

is indicative of the difficulty of the complete problem which should take into account both pressure and shear waves. Similar problems are discussed in an article by Censor<sup>11</sup> in which a piecewise linear approximation is considered.

In view of the foregoing summary, we are motivated to investigate the effect of a nonuniformly moving Newtonian fluid layer, situated between moving elastic media, on the transmission, reflection, and scattering of elastic waves. Not surprisingly, the general formulation for waves in such a medium (whose mean velocity and pressure fields are prescribed by the solution of the relevant Navier-Stokes equations) are much more complicated than the case considered by Graham and Graham.<sup>10</sup>

As a first step we consider problems restricted to the propagation of shear waves in two-dimensional flows, such that no compressional waves are produced. In a sense, this problem is complementary to the case of Graham and Graham,<sup>10</sup> who considered compressional waves only. As far as we know, this is the first attempt to consider shear waves in nonuniformly moving media. Although the present problem is only a special case, it leads to better understanding of the adequate interface conditions between relatively moving elastic media as discussed in Refs. 7 and 8.

The two fundamental configurations discussed subsequently—(a) flow between plane parallel surfaces and, (b) flow between two concentric cylindrical surfaces—yield closed form solutions. Hence it is convenient to discuss various limiting cases such as vanishingly small thickness and to examine the impli-

cations of the results. It will be seen that this study indicates that appropriate boundary conditions for the scattering of waves across an interface between relatively moving elastic media should be continuity of stress and perturbation velocity, and not displacement. This is not in conflict with the existing theory for acoustical (shearless) waves. However, it indicates that the problem of contact between sliding elastic media is not perfectly clear yet, and probably there is more than one set of boundary conditions, depending on the mathematical modeling of the physical situation.

From the point of view of applications the problem is of interest for problems of lubrication (especially the cylindrical case) and for measurements of sliding or rotating machinery which can be probed by means of mechanical waves but is not otherwise accessible.

I. PROPAGATION OF SHEAR WAVES IN VISCOUS FLOW

We wish to study the propagation of a small perturbing shear wave whose displacement is in the *z* direction in a viscous fluid whose mean velocity field *V* is independent of *z*. For a perturbing velocity given by *vz* the condition that dilation be zero requires that *v*=*v*(*x*,*y*,*t*). The mean flow must satisfy the Navier-Stokes equation

$$\rho \left( \frac{\partial}{\partial t} \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla P + \mu \nabla^2 \mathbf{V} + \left( \eta + \frac{\mu}{3} \right) \nabla (\nabla \cdot \mathbf{V}) \quad (1)$$

and the equation of continuity

$$\partial \rho / \partial t + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = 0. \quad (2)$$

As usual  $\mu$  and  $\eta$  are the shear and bulk viscosities, respectively,  $\rho$  is the density, and *P* is the pressure. We now allow the velocity *V* to be perturbed by the shear wave *vz* (which cannot affect density or pressure). In order to obtain equations on *v*, which is assumed small enough so that terms of higher order than first are negligible, substitute *V*+*vz* into Eqs. 1 and 2. All equations are identically satisfied except the Navier-Stokes equations in the *z* direction, which is

$$\rho \left( \frac{\partial v}{\partial t} + V_x \frac{\partial v}{\partial x} + V_y \frac{\partial v}{\partial y} \right) = \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \quad (3)$$

It is immediately seen that *v* is independent of *V<sub>z</sub>*, the *z* component of the mean flow, and henceforth *V<sub>z</sub>* shall be disregarded.

We shall consider the propagation of such a shear wave for two cases of steady mean flow.

A. Problem 1: Flow between Plane Parallel Surfaces

In an inertial rectangular Cartesian coordinate system, assume that the region *y*<0 is occupied by an

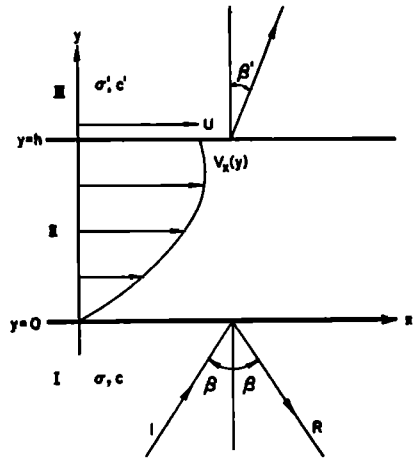


FIG. 1. Geometry for problem 1.

elastic medium at rest (medium I), characterized by a shear-wave speed *c* and density  $\sigma$ . The region *y*>*h* is occupied by another elastic medium (medium III), characterized by *c'*,  $\sigma'$ , moving with velocity *Ux*, relative to medium I. The region 0<*y*<*h* contains a viscous fluid (medium II), and its mean motion under a constant pressure gradient in the *x* direction is

$$\mathbf{V} = V_x(y) \hat{x} = [(U+C)(y/h) - C(y/h)^2] \hat{x}. \quad (4)$$

(See Fig. 1.) Note that the constant *C* is proportional to the imposed pressure gradient.

In medium I, plane harmonic shear waves are assumed to propagate in directions lying in the *xy* plane. The displacement is in the *z* direction. These *SH* waves have a dependence on *x* and *t* given by  $\exp(ik_x x - i\omega t)$ . Due to subsequent boundary conditions this factor is invariant everywhere to an observer in the laboratory frame of reference.

Thus in medium II the perturbation *v<sub>II</sub>*(*x*,*y*,*t*)*z* must be of the form

$$v_{II}(x,y,t) = f(y) \exp(ik_x x - i\omega t). \quad (5)$$

The shear stress associated with this velocity is  $\tau_{yz}$ , which equals  $\mu \partial v / \partial y$ . Substitution of Eqs. 4 and 5 into Eq. 3 yields

$$\frac{d^2 f}{dy^2} + (a_0 + a_1 y + a_2 y^2) f = 0, \quad (6)$$

$$a_0 = i\omega\rho/\mu - k_x^2,$$

$$a_1 = -ik_x\rho(U+C)/\mu h,$$

$$a_2 = ik_x\rho C/\mu h^2.$$

The general solution of Eq. 6 is available in terms of confluent hypergeometric functions or parabolic cylinder functions (see, for example, Murphy<sup>12</sup>). For the special case *C*≡0 we have the case of a Couette flow. For this

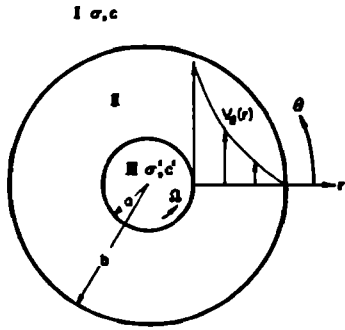


FIG. 2. Geometry for problem 2.

ciated with a variable density  $\rho(r)$ ; hence  $p_m, q_m$  are functions of  $r$ . However for subsonic flow  $\rho$  can usually be taken as constant, see Appendix A. Equation 10 has the general solution

$$f_m = A_m H_{q_m}^{(1)}(p_m r) + B_m H_{q_m}^{(2)}(p_m r), \quad (11)$$

where  $H_{q_m}^{(1)}$  and  $H_{q_m}^{(2)}$  are Hankel functions of the first and second kind, respectively, of complex order  $q_m$  and complex argument  $p_m r$ . We choose the root  $q_m$  so that as  $\Omega \rightarrow 0, q_m \rightarrow m$ . Exploiting the orthogonality of the functions  $e^{im\theta}$  in Eq. 9 and the similar expressions given subsequently for regions I and III, we can solve for each mode independently, in the same manner as in problem 1.

II. THE SCATTERING PROBLEM

A. Problem 1

In medium I, the incident wave (assumed to be of unit amplitude) and the reflected wave are specified by

$$u_I = \exp(ik_x x + ik_y y - i\omega t) + R \exp(ik_x x - ik_y y - i\omega t), \quad (12)$$

$$(k_x^2 + k_y^2)^{1/2} = \omega/c,$$

where  $u$  is the displacement in the  $z$  direction and the incident wave propagates in a direction specified by  $k_x = (\omega/c) \sin\beta$  (see Fig. 1). The velocity  $v_I$  in the  $z$  direction is  $-i\omega u_I$ . Such a displacement field gives rise to a shear stress  $\tau_{yz} = \sigma^2 \partial u_I / \partial y$ .

In medium III moving with velocity  $U \hat{x}$  the transmitted wave is given by

$$u_{III} = T \exp(ik_x x + ik'_y y - i\omega t), \quad (13)$$

$$(k_x^2 + k'_y{}^2)^{1/2} = \omega/c'_{\text{eff}},$$

$$c'_{\text{eff}} = c' / (1 - k_x U / \omega),$$

derived by applying the pertinent Galilean transformations and Snell's law, i.e., the invariance of  $\exp(ik_x x - i\omega t)$ . Hence the angle of propagation in medium III is given by  $k_x = (\omega/c'_{\text{eff}}) \sin\beta'$ . Similarly in medium I, the shear stress is given by  $\sigma' c'^2 \partial u / \partial y$ . However, to compute the velocity of medium III in the  $z$  direction, it is necessary to take the material derivative, which to first order in  $u$  yields

$$v_{III} = \frac{\partial u_{III}}{\partial t} + U \frac{\partial u_{III}}{\partial x} = -i\omega(1 - k_x U / \omega) u_{III}. \quad (14)$$

The boundary conditions as stated above are that velocity and traction be continuous at  $y=0, y=h$ . The velocity conditions give

$$-i\omega(1 + R) = Y_0^3 [A J_3(\frac{2}{3} Y_0^3) + B J_{-3}(\frac{2}{3} Y_0^3)] \equiv F(Y_0),$$

$$-i\omega(1 - k_x U / \omega) e^{ik'_y h} T = F(Y_h),$$

$$Y_0 \equiv Y(0) = a_0 a_1^{-1},$$

$$Y_h \equiv Y(h) = a_0 a_1^{-1} + a_1^3 h. \quad (15)$$

case  $a_2 \equiv 0$ , and the general solution of Eq. 6 becomes,<sup>13</sup>

$$f = V^3 [A J_{\frac{1}{3}}(\frac{2}{3} V^3) + B J_{-\frac{1}{3}}(\frac{2}{3} V^3)], \quad (7)$$

$$Y = a_0 a_1^{-1} + a_1^3 y,$$

where  $J$  denotes the Bessel function. The constants  $A, B$ , together with a reflection coefficient  $R$  for the reflected wave in medium I, and the transmission coefficient  $T$  in medium III will be determined from conditions of continuity of velocity and traction across  $y=0, y=h$ .

B. Problem 2: Flow between Concentric Cylindrical Surfaces

In an inertial cylindrical coordinate system  $r, \theta, z$  the region  $r > b$  is occupied by an elastic medium at rest (medium I). The region  $r < a$  is occupied by another elastic medium (III), rotating with an angular velocity  $\Omega$ . Separating the rotating shaft from the external medium is a viscous fluid (II), whose mean motion is (see, for example, Charlton<sup>14</sup>)

$$V = V_\theta(r) \hat{\theta} = -\frac{\Omega r}{(b/a)^2 - 1} \left[ \left(\frac{b}{r}\right)^2 - 1 \right] \hat{\theta} \quad (8)$$

(see Fig. 2). Solutions of Eq. 3 (in cylindrical coordinates) must be periodic in  $\theta$ ; for harmonic time variation  $e^{-i\omega t}$  the perturbation  $v_{II}(r, \theta, t) \hat{z}$  assumes the form

$$v_{II} = e^{-i\omega t} \sum_{m=-\infty}^{\infty} f_m(r) e^{im\theta}. \quad (9)$$

Substitution of Eqs. 8 and 9 into Eq. 3 yields

$$\frac{d^2 f_m}{dr^2} + \frac{1}{r} \frac{df_m}{dr} + \left( p_m^2 - \frac{q_m^2}{r^2} \right) f_m = 0,$$

$$p_m^2 = \frac{i\omega\rho}{\mu} \left( 1 + \frac{m}{(b/a)^2 - 1} \frac{\Omega}{\omega} \right), \quad (10)$$

$$q_m^2 = m^2 + \frac{i\rho m \Omega b^2 / \mu}{(b/a)^2 - 1}.$$

For a compressible flow, the flow field Eq. 8 is asso-

The continuity of traction yields

$$\sigma c^2 i k_y (1-R) = \mu a_1^3 \left\{ \frac{1}{2} Y_0^{-1} [A J_{\frac{1}{2}}(\frac{2}{3} Y_0^3) + B J_{-\frac{1}{2}}(\frac{2}{3} Y_0^3)] + Y_0 [A J'_{\frac{1}{2}}(\frac{2}{3} Y_0^3) + B J'_{-\frac{1}{2}}(\frac{2}{3} Y_0^3)] \right\} = G(Y_0), \quad (16)$$

$$\sigma' c'^2 i k'_y e^{i k' y' h} T = G(Y_h),$$

where the primes on the Bessel functions denote differentiation with respect to the argument. Thus the solution is formally complete.

**B. Problem 2**

In medium I the displacement is  $u_I = u_i + u_s$ , where the incident wave  $u_i$  is given by

$$u_i = e^{i k x - i \omega t} = e^{-i \omega t} \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{i m \theta}, \quad k = \omega/c. \quad (17)$$

The scattered wave  $u_s$  is chosen as

$$u_s = e^{-i \omega t} \sum_{m=-\infty}^{\infty} S_m H_m^{(1)}(kr) e^{i m \theta} \quad (18)$$

in order that it satisfy the Sommerfeld<sup>15</sup> radiation condition. The velocity is  $-i\omega u_I$ , and the stress  $\tau_{rz}$  is given by  $\sigma c^2 \partial u_I / \partial r$ . In the rotating cylinder, medium III, the displacement will consist of a radial component due to the rotation, and the  $z$  component due to the shear wave. In the equations of motion in a coordinate system rotating with the medium these components will be uncoupled. The equation governing  $u_{III}$ , the  $z$  component, transformed into the laboratory coordinate system, is

$$c'^2 \nabla^2 u_{III} = \left( \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta} \right)^2 u_{III}. \quad (19)$$

Letting  $u_{III}$  be represented by

$$u_{III} = e^{-i \omega t} \sum_{m=-\infty}^{\infty} g_m(r) e^{i m \theta} \quad (20)$$

yields the ordinary differential equation satisfied by the functions  $g_m$ :

$$\frac{d^2 g_m}{dr^2} + \frac{1}{r} \frac{dg_m}{dr} + \left( k_m^2 - \frac{m^2}{r^2} \right) g_m = 0, \quad (21)$$

$$k_m = \frac{\omega}{c'} \left( 1 - \frac{m\Omega}{\omega} \right).$$

Subject to the condition that  $u_{III}$  is finite on the axis  $r=0$ , the solution of Eq. 21 is

$$g_m = T_m J_m(k_m r). \quad (22)$$

$S_m, T_m$  are constants, to be determined by boundary conditions. The stress in medium III is again given by  $\sigma' c'^2 \partial u_{III} / \partial r$ , but to get the velocity  $v_{III}$ , we have to

use the material derivative, which to first order in  $u_{III}$  is

$$v_{III} = \frac{\partial u_{III}}{\partial t} + \Omega \frac{\partial u_{III}}{\partial \theta} = -i\omega \left( 1 - \frac{m\Omega}{\omega} \right) u_{III}. \quad (23)$$

The continuity of the velocity at  $r=b$  and  $r=a$  thus gives for each mode  $m$

$$-i\omega [i^m J_m(kb) + S_m H_m^{(1)}(kb)] = A_m H_{q_m}^{(1)}(p_m b) + B_m H_{q_m}^{(2)}(p_m b),$$

$$-i\omega \left( 1 - \frac{m\Omega}{\omega} \right) T_m J_m(k_m a) = A_m H_{q_m}^{(1)}(p_m a) + B_m H_{q_m}^{(2)}(p_m a). \quad (24)$$

The continuity of the traction across  $r=b, r=a$  yields

$$\sigma c^2 k [i^m J'_m(kb) + S_m H_m^{(1)'}(kb)] = \mu p_m [A_m H_{q_m}^{(1)'}(p_m b) + B_m H_{q_m}^{(2)'}(p_m b)],$$

$$\sigma' c'^2 k_m T_m J'_m(k_m a) = \mu p_m [A_m H_{q_m}^{(1)'}(p_m a) + B_m H_{q_m}^{(2)'}(p_m a)]. \quad (25)$$

Consequently the problem is formally solved; the coefficients  $S_m, A_m, B_m, T_m$  are found from Eqs. 24 and 25 and substituted in the relevant Fourier series above.

**III. DISCUSSION**

Inasmuch as we have analytical results available, various limiting cases of interest can be discussed.

Consider the case of fluid layers thin compared to the wavelengths. For the case of the plane layer, the constants  $R, T, A$ , and  $B$  are obtained from Eqs. 15 and 16. Expanding  $R, T$  as a Maclaren series in the dimensionless quantity  $k_z h$ , yields, for the leading terms

$$R = \frac{\sigma c \cos \beta - \sigma' c' \cos \beta'}{\sigma c \cos \beta + \sigma' c' \cos \beta'} + O(k_z h),$$

$$T = \frac{2\sigma c \cos \beta / (1 - U k_z / \omega)}{\sigma c \cos \beta + \sigma' c' \cos \beta'} + O(k_z h). \quad (26)$$

The same limiting values are obtained in a less rigorous way by considering  $h \rightarrow 0$  and identifying the right-hand sides of the two equations of Eq. 15 pertaining to the fluid layer; similarly for Eq. 16. Thus we would have the following two equations for  $R$  and  $T$ :

$$v_I = -i\omega u_I = -i\omega (1 - k_z U / \omega) u_{III} = v_{III},$$

$$\tau_{yzi} = \sigma c^2 \frac{\partial u_I}{\partial y} = \sigma' c'^2 \frac{\partial u_{III}}{\partial y} = \tau_{yziII}. \quad (27)$$

For the case of two elastic half-spaces sliding relative to one another, the exact type of contact is unknown. However, under the assumptions that there is some sort of viscous layer which fills the space between the

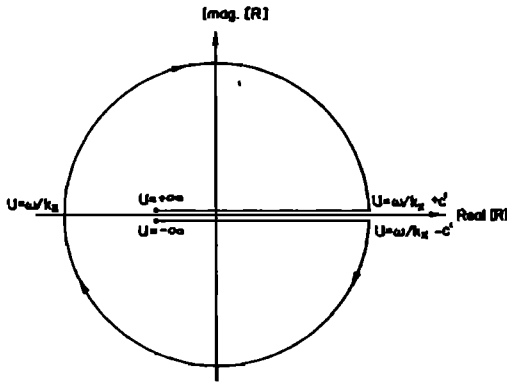


FIG. 3. Locus of  $R$  as a function of  $U$  for the case  $\sigma'c' \geq \sigma c \cos\beta$ .

half-spaces (not necessarily Newtonian) which has a smooth mean velocity profile and a perturbation velocity  $v_{II}$  of the form of Eq. 5 with  $f(y)$  continuous, and that the usual boundary condition, that the velocity of the fluid relative to the wall is zero at every point, it is expected from the procedure leading to Eqs. 27 that in the limit as  $h \rightarrow 0$ , Eqs. 27 will hold. This implies that, under the above assumptions, for a layer thin enough to neglect, the interface conditions for this problem are the continuity of the perturbation velocity and the shear stress. This has been proved rigorously, however, only for a Newtonian fluid by comparing the limiting values of  $R$  and  $T$  as given by Eqs. 26 with the values that Eqs. 27 give. It is interesting to note that in the acoustics of moving media, the studies of Miles<sup>1</sup> and Ribner<sup>2</sup> point to the fact that the displacement, rather than the velocity, should be continuous (see also Refs. 3-5 and 16). This is not contradicted by the present study, as the viscous model for the interface as used above does not seem to be appropriate in the purely acoustical problem.

Some special cases of interest are derived from 26. Consider the case of perfect reflection  $|R|=1$ . For  $R=1$  the condition is  $U+c'=\omega/k_x \equiv c/\sin\beta$ . From Snell's law this value of  $\beta$  is the critical angle for which  $\sin\beta'=1$ . This is analogous to the case when there is no motion. As  $U$  becomes larger, criticality is maintained with  $|R|=1$  and a varying phase, until  $R=-1$  at  $U=\omega/k_x$ , and  $c'_{eff} \rightarrow \infty$ . For larger  $U$ ,  $|R|=1$  continues and  $c'_{eff}$  is negative until  $U-c'=\omega/k_x$  and  $R=1$ . For even larger  $U$  once again  $\beta'$  is real and waves propagate into region III. It is easily seen that for  $\sigma'c' \geq \sigma c \cos\beta$  there are two velocities  $U$  for which  $R=0$ , given by

$$\left(1 - \frac{k_x U}{\omega}\right)^2 = \frac{(c'k_x/\omega)^2}{1 - (\sigma c'/\sigma'c)^2 \cos^2\beta} \quad (28)$$

The behavior of  $R$  as a function of  $U$  is sketched in Fig. 3, for the case where  $\sigma'c' \geq \sigma c \cos\beta$ . Note that  $\text{Re}[R]$  is symmetrical about the value  $U=\omega/k_a$  and  $\text{Im}[R]$  is antisymmetrical.

Similarly  $A$  and  $B$  and thus  $f(7)$  can be calculated and expanded in orders of  $k_x h$ . In the limit this yields  $v_{II} = v_I = v_{III}$  at  $y=0$ , as expected.

For problem 2, for a vanishingly thin layer  $a \rightarrow b$  we get, from the limit of Eqs. 24 and 25

$$S_m = -i^m \frac{\sigma c J_m(k_m b) J_m'(k b) - \sigma' c' J_m'(k_m b) J_m(k b)}{\sigma c J_m(k_m b) H_m^{(1)'}(k b) - \sigma' c' J_m'(k_m b) H_m^{(1)}(k b)}$$

$$T_m = \frac{i^m \sigma c}{(1 - m\Omega/\omega)} \times \frac{J_m(k b) H_m^{(1)'}(k b) - J_m'(k b) H_m^{(1)}(k b)}{\sigma c J_m(k_m b) H_m^{(1)'}(k b) - \sigma' c' J_m'(k_m b) H_m^{(1)}(k b)} \quad (29)$$

For the particular case where  $\omega/\Omega = M$ , an integer, we get

$$S_M = -i^M J_M(k b) / H_M^{(1)}(k b), \quad (30)$$

which is the coefficient obtained for scattering by a rigid cylinder. Note that as  $\omega/\Omega \rightarrow M$ ,  $T_M \rightarrow \infty$  as  $(k_M)^{-1M}$ , but  $T_M J_M(k_M r)$  is finite, and proportional to  $r^{1M}$ . This agrees with the fact that the solution of Eq. 21 for  $k_M=0$  is  $r^{1M}$ . Consequently for mode  $M$  there is a finite displacement in medium III; however, in the rotating coordinate system it is time-independent. This is clear from the fact that in the rotating system we measure  $\theta' = \theta - \Omega t$ ; therefore

$$\exp(iM\theta - i\omega t) = \exp(iM\theta - i\Omega M t) \equiv \exp(iM\theta').$$

Hence no waves propagate in medium III; this explains the behavior of this mode as that of a rigid cylinder (Eq. 30).

For the mode  $m=0$  it is seen that the rotation has no effect since  $m\Omega$  appears as a factor. Hence, for identical media,  $S_0=0$  in Eq. 29 and for small  $\Omega$  the dipole term  $m=\pm 1$  will predominate.

This problem has been conceived as a first step in the full study of the transmission and reflection, and scattering of elastic waves in the presence of viscous lubricating layers between relatively moving elastic media. The considerably more difficult plane strain problem involving both compression and shear waves will be discussed in subsequent papers and will provide results in the limit, directly comparable with those obtained by previous researchers.

APPENDIX A

In problem 2 the change of density as a function of  $r$  was neglected. It is shown here that for a wide range of parameters this is justified.

Assuming  $V = V_0(r)\hat{\theta}$ ,  $P = P(r)$ , and  $\rho = \rho(r)$  and substituting the Navier-Stokes equation (1) into cylindrical coordinates gives (noting that  $\nabla \cdot V = 0$

everywhere):

$$\rho(r) \frac{V_0^2}{r} = \frac{dP}{dr} \tag{A1}$$

and

$$\frac{d^2 V_\theta}{dr^2} + \frac{1}{r} \frac{dV_\theta}{dr} - \frac{V_\theta}{r^2} = 0. \tag{A2}$$

Subject to the boundary conditions,  $V_\theta(b)=0$ ,  $V_\theta(a)=\Omega d$ , Eq. A2 yields for  $V_\theta$  the solution given by Eq. 8. Let the fluid be elastically compressible, i.e.,  $dP/d\rho=C_p^2$ ; hence

$$P(r) - P(b) = C_p^2 [\rho(r) - \rho(b)], \tag{A3}$$

where  $C_p$  is the sound speed in the fluid, assumed constant over the range of pressures encountered in the layer. Then Eqs. A1 and A3 yield a solution for  $\rho(r)$ :

$$\frac{\rho(r)}{\rho(b)} = \exp \left\{ -\frac{1}{2} \left( \frac{\Omega b}{C_p [(b/a)^2 - 1]} \right)^2 \times \left[ \left( \frac{b}{r} \right)^2 - \left( \frac{r}{b} \right)^2 - 4 \ln \frac{b}{r} \right] \right\}. \tag{A4}$$

Note that for  $r < b$ ,  $\rho(r)/\rho(b)$  is a positive, monotonically increasing function of  $r/b$  which approaches 1 as  $r$  approaches  $b$  with a slope of 0. Thus the greatest density variation occurs at  $r=a$ , and this is given by

$$\begin{aligned} \frac{\rho(a)}{\rho(b)} &= \exp \left\{ -\frac{1}{2} \left( \frac{\Omega b}{C_p [(b/a)^2 - 1]} \right)^2 \times \left[ \left( \frac{b}{a} \right)^2 - \left( \frac{a}{b} \right)^2 - 4 \ln \frac{b}{a} \right] \right\} \\ &= \exp \left\{ \frac{a^2 \Omega^2 [1 - \alpha^4 + 4\alpha^2 \ln(\alpha)]}{2C_p^2 (1 - \alpha^2)^2} \right\}, \end{aligned} \tag{A5}$$

where  $\alpha = a/b < 1$ . The greatest variation of density,

holding  $a\Omega/C_p$  constant occurs for  $\alpha \rightarrow 0$  and it is given by  $\rho(a)/\rho(b) \sim \exp(-a^2\Omega^2/2C_p^2)$ . Thus to ensure that the density variation in the unperturbed flow is 1% or less we are restricted to velocities at the inner wall of less than 15% of the fluid sound speed, and even for wall velocity of 45% of the sound speed, the density variation is less than 10%, for the worst possible situation of  $b/a$  very large.

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<sup>13</sup> A. Sommerfeld, *Partial Differential Equations in Physics* (Academic, New York, 1964), p. 188 ff.

<sup>14</sup> F. Charlton, *Textbook of Fluid Dynamics* (Van Nostrand, London, 1967).

<sup>15</sup> Ref. 13, p. 314, Eq. 34.

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