Acoustical Doppler effect analysis—Is it a valid method?[a)]

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The Doppler effect approach is a powerful tool used in various areas of physics to analyze motion in a wave field, by observing the frequency shifts of scattered waves. In acoustics, Doppler effect analysis is used for diagnostics in biomedical systems and for industrial applications, e.g., in cases where scatterers are present in moving fluids. The theoretical models used to analyze such systems usually ignore the fact that the moving ambient medium might produce effects of the same order of magnitude as the scatterers themselves. It has even been argued that in certain circumstances such effects should completely cancel the Doppler frequency shifts. A model is developed here that contributes to our understanding of the scattering in the presence of moving objects and space- and time-dependent moving media. The model is restricted to irrotational flows, neglects velocity effects except of the first order in the Mach number v/c, and assumes slow variations in the ambient medium. These restrictions facilitate the analytical discussion of specific canonical problems. The present study indicates that Doppler effects can be produced by moving scatterers in a medium at rest, scatterers at rest in a moving medium, and in configurations in which combined motion of scatterers and media take place. The Doppler effects are of the same order of magnitude in all cases. This vindicates the simple models used for research and applications, which assume that the moving objects produce Doppler effects but neglect the flow of the surrounding medium.

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INTRODUCTION

Recently Piquette and Van Buren[1,2] criticized a model proposed by Censor,[3] analyzing scattering from time varying obstacles. An earlier article by Censor[4] has been criticized by Rogers.[5,6] Essentially, the objections were concerned with the facts that a linear medium has been assumed, and that the motion imparted to the medium by the moving scatterers has been ignored. It is interesting to note, as observed by Toman,[7] in a historical review of the Doppler effect, that objections to Doppler's theory had been raised by Petzval shortly after the appearance of Doppler's original work.[8] His arguments concerned the motional effects due to the moving media.

It is therefore an interesting problem of fundamental importance, to analyze the acoustical scattering problem involving time-dependent boundaries, taking into account the medium motion as well. The general problem is too complicated; hence, presently, some restrictions are incorporated. The main idea of the present study is to present the wave field as a product of the velocity-independent solution, and a factor that is essentially a WKB approximation, which takes into account the space and time velocity effects. Such a solution is essentially a ray (as opposed to wave) formalism, and has to be appropriately qualified. The present model applies to irrotational flows satisfying

$$\nabla \times \mathbf{v}(\mathbf{r},t) = 0\,, \tag{1}$$

where $\mathbf{v}$ is the space-time-dependent velocity of the ambient medium. Moreover, only effects of first order in the Mach number are retained. Finally, we have to assume slow variations of $\mathbf{v}(\mathbf{r},t)$, relative to distances and time intervals of the order of the wavelength and the period, respectively, of the sound wave in question.

The theory is applied to a few canonical problems in one and three dimensions. The results indicate that, at least in systems satisfying the present restrictions, the medium effects are of the same order of magnitude as the effects produced by the scatterers when the medium effects are ignored. Strictly speaking, the medium's effects and the scatterers' boundary effects should be considered simultaneously. However, it is shown that the former become negligibly small as the distance of the observer from the object increases, while the latter effect is independent of distance. Consequently, as an approximation, models that ignore the motion imparted by the surface to the surrounding medium are still correct.

I. THEORY

The wave equation for the acoustical potential is given by

$$[\nabla'^2 - (1/c^2)\partial_{t'}^2]\psi'(\mathbf{r}',t') = 0\,, \tag{2}$$

where $\partial'_t = \partial/\partial t'$ and the prime indicates that the observer is attached to a region where the medium is at rest, i.e., he is "comoving" with the medium according to the bulk velocity $\mathbf{v}(\mathbf{r},t)$. The bulk velocity is oberved from the "laboratory"

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frame of reference, and expressed in terms of its appropriate space-time coordinates $\mathbf{r},t$. It is assumed that $\mathbf{v}$ changes slowly over distances of the order of wavelength, and time intervals of the order of the period, of the field ψ' in question. This assumption on the bulk velocity is already restricting the model and qualifying it for the WKB approximation introduced below.

We are interested in $\psi(\mathbf{r},t)$, the field measured by the observer in the laboratory frame of reference. It is assumed that

$$\psi'(\mathbf{r}',t') = \psi(\mathbf{r},t) \tag{3}$$

is an invariant; i.e., if we have a transformation $\mathbf{r}' = \mathbf{r}'(\mathbf{r},t)$, $t' = t'(\mathbf{r},t)$, then $\psi'(\mathbf{r}'[\mathbf{r},t],t'[\mathbf{r},t]) = \psi(\mathbf{r},t)$. Unlike electromagnetic theory, in acoustics it is safe to assume the Galilean transformation,

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t', \quad t = t', \tag{4}$$

for constant velocities $\mathbf{v}$. For variable velocities $\mathbf{v}(\mathbf{r},t)$ it is meaningless to use (4), although this is often done.[9] A logical generalization of (4) would be

$$\mathbf{r} = \mathbf{r}' + \int_0^{t'} \mathbf{v}(\bar{\mathbf{r}},\bar{t})d\bar{t}, \tag{5}$$

where $\bar{\mathbf{r}}, \bar{t}$ denote integration variables. Using the chain rule of calculus (5) yields

$$\frac{\partial}{\partial \mathbf{r}'} = \frac{\partial}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial \mathbf{r}'} + \frac{\partial}{\partial t}\frac{\partial t}{\partial \mathbf{r}'}, \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t}\frac{\partial t}{\partial t'} + \frac{\partial}{\partial \mathbf{r}}\frac{\partial \mathbf{r}}{\partial t'}, \tag{6}$$

where $\partial/\partial\mathbf{r}$ denotes ∇; hence, (6) becomes

$$\nabla' = \nabla, \quad \partial_{t'} = \partial_t + \mathbf{v}\cdot\nabla. \tag{7}$$

Fluid mechanics texts usually refer to $\partial_{t'}$ as the total derivative, the moving derivative, or the material derivative. In (7), $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$ is the local instantaneous velocity in the laboratory frame of reference.

In terms of (7), we obtain (2), subject to (3),

$$[\nabla^2 - (1/c^2)(\partial_t + \mathbf{v}\cdot\nabla)^2]\psi(\mathbf{r},t) = 0 \tag{8}$$

and to the first order in v/c, the Mach number, we have

$$[\nabla^2 - (1/c^2)(\partial_t^2 + 2\mathbf{v}\cdot\nabla\,\partial_t)]\psi(\mathbf{r},t) = 0. \tag{9}$$

Except for trivial cases, solutions of (9) are not readily available. Acoustics and electrodynamics in moving media, where $\mathbf{v} = \mathbf{v}(\mathbf{r})$ was only space-dependent, have been considered previously.[10,11] Essentially, the approximations used for these problems involved WKB-type exponentials with line integral exponents. The logical extension for taking into account temporal variations is to use a four-dimensional line integral, of the kind used in the Hamiltonian ray theory.[12–14] Accordingly we make the ansatz

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r})e^{-i\omega_0 t + i\beta(\mathbf{r},t)}, \qquad \beta(\mathbf{r},t) = \int_{\mathbf{r}_0,t_0}^{\mathbf{r},t} [g(\bar{\mathbf{r}},\bar{t})\cdot d\bar{\mathbf{r}} - h(\bar{\mathbf{r}},\bar{t})d\bar{t}], \tag{10}$$

where $\psi_0 e^{-i\omega_0 t}$ is the solution of (9) for $\mathbf{v} = 0$ and β is a four-dimensional line integral and provides the correction due to the velocity in (9). Hence $\mathbf{g}, h$ are of first order in v/c. We are only able to deal with (10) if the integral depends on the limits and is independent of the path of integration; i.e., $\beta = \int d\beta$, where $d\beta = \mathbf{g}\cdot d\bar{\mathbf{r}} - h\,d\bar{t}$ is a total differential. This is equivalent to having

$$\nabla\times\mathbf{g} = 0, \quad \frac{\partial\mathbf{g}}{\partial t} + \nabla h = 0. \tag{11}$$

The first expression (11) is Snell's law in disguise; the second one is the extension to the case of four-dimensional line integrals.[15] It follows from (10), (11) that

$$\mathbf{g} = \nabla\beta, \quad h = -\frac{\partial\beta}{\partial t} \tag{12}$$

according to the rule of differentiating an integral with respect to the upper limit. Substituting (10) into (9) and keeping only first-order v/c terms yield

$$e^{-i\omega_0 t + i\beta}[(\nabla^2 + \omega_0^2/c^2)\psi_0(\mathbf{r}) + i2\nabla\psi_0\cdot\mathbf{g} + i\psi_0\nabla\cdot\mathbf{g} + 2\omega_0 h\psi_0/c^2 + i2\omega_0\mathbf{v}\cdot\nabla\psi_0/c^2] = 0. \tag{13}$$

Since $(\nabla^2 + \omega_0^2/c^2)\psi_0(\mathbf{r}) = 0$ by definition, we have to choose $\mathbf{g},h$ such that the sum of the remaining terms in brackets (13) vanishes. For time-independent media we have $h = 0$ identically. If we assume $\mathbf{g} = -\omega_0\mathbf{v}/c^2$, then (13) vanishes for media satisfying $\nabla\cdot\mathbf{v} = 0$, i.e., for homogeneous incompressible fluids.[16] For this class of problems, (10) reduces to

$$\psi = \psi_0(\mathbf{r})\exp\left[-i\omega_0\left(t + \frac{1}{c_2}\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v}\cdot d\bar{\mathbf{r}}\right)\right] \tag{14}$$

and for a known function $\mathbf{v}$, (14) can be evaluated. Presently $\mathbf{v}(\mathbf{r},t)$ is time-dependent and the simple result (14) does not apply.

Applying ∂_t to the expression in brackets in (13) and exploiting (11), we obtain

$$2\nabla\psi_0\cdot\nabla h + \psi_0\nabla^2 h + i2\omega_0\frac{\partial h}{\partial t}\frac{\psi_0}{c^2} - 2\omega_0\frac{\partial\mathbf{v}}{\partial t}\cdot\frac{\nabla\psi_0}{c^2} = 0. \tag{15}$$

This is a differential equation on $h(\mathbf{r},t)$, and in general not simpler than the one with which we started, (9). However, in special circumstances (15) will become simpler. The first simplifying assumption will be that

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}(\mathbf{r})e^{-i\Omega t}; \tag{16}$$

i.e., the velocity will be considered for harmonic time variation at frequency Ω. Consequently $\mathbf{g},h$ will also be time harmonic, having the same frequency. Consequently (15) is rewritten as

$$2\nabla\psi_0\cdot\nabla h + \psi_0\nabla^2 h + 2\omega_0\Omega\, h\psi_0/c^2 + i2\omega_0\Omega\mathbf{v}\cdot\nabla\psi_0/c^2 = 0 \tag{17}$$

and once h is derived, $\mathbf{g}$ follows from (11),

$$\mathbf{g} = -i\nabla h/\Omega, \tag{18}$$

and it is noted that $\nabla\times\mathbf{g} = 0$ is identically satisfied. Once $\mathbf{g},h$ are found, the problem has been solved and ψ in (10) is known.

II. PLANE WAVE PROPAGATION

In order to gain insight into the features of the above model, consider

$$\psi_0 = e^{i\mathbf{k}_0\cdot\mathbf{r}}, \quad k_0 = \omega_0/c\,; \tag{19}$$

i.e., in the absence of medium bulk velocity we are dealing with a plane wave. For this case (17) becomes

$$i2\mathbf{k}_0\cdot\nabla h + \nabla^2 h + 2\omega_0\Omega h/c^2 - 2\omega_0\Omega\mathbf{v}\cdot\mathbf{k}_0/c^2\,. \tag{20}$$

Further, assume $\mathbf{v},h$ to be low-frequency plane waves,

$$\mathbf{v} = \mathbf{v}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\Omega t}, \quad h = h_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\Omega t}, \quad K = \Omega/C\,, \tag{21}$$

where $\mathbf{v}_0$, h_0 are constants and C is the appropriate phase velocity, which due to dispersion might be different from c. Substituting (21) in (20) yields

$$-2\mathbf{k}_0\cdot\mathbf{K}h - K^2h + 2\omega_0\Omega h/c^2 - 2\omega_0\Omega\mathbf{v}\cdot\mathbf{k}_0/c^2 \tag{22}$$

and taking the real part of (22) (note that all operations thus far are linear, hence taking the real part is a valid option), we find

$$h = h_0\cos(\mathbf{K}\cdot\mathbf{r} - \Omega t),$$
$$h_0 \frac{2\omega_0\Omega\mathbf{v}_0\cdot\mathbf{k}_0/c^2}{2\omega_0\Omega/c^2 - K^2 - 2\mathbf{k}_0\cdot\mathbf{K}}, \tag{23}$$
$$\mathbf{g} = \mathbf{K}h/\Omega\,.$$

Having computed $\mathbf{g},h$, the velocity-dependent solution is available,

$$\psi = e^{i\mathbf{k}_0\cdot\mathbf{r} - i\omega_0 t + i\beta},$$
$$\beta = h_0\int_{\mathbf{r}_0,t_0}^{\mathbf{r},t}\left(\frac{\mathbf{K}}{\Omega}\cdot d\,\bar{\mathbf{r}} - d\,\bar{t}\right)\cos(\mathbf{K}\cdot\bar{\mathbf{r}} - \Omega\bar{t})\,, \tag{24}$$

where the integral must be evaluated, in order to obtain explicit expressions for ψ. For special cases, this is performed in Sec. III. For the lower limit of β taken as $\mathbf{r}_0 = 0$, $t_0 = 0$, (24) can be recast in the form

$$\gamma = \int_{0,0}^{r,t}\left[d\,\bar{\mathbf{r}}\cdot\left(\mathbf{k}_0 + \frac{h_0\mathbf{K}}{\Omega}\cos(\mathbf{K}\cdot\bar{\mathbf{r}} - \Omega\bar{t})\right)\right.$$
$$\left. - d\,\bar{t}\,[\omega_0 + h_0\cos(\mathbf{K}\cdot\bar{\mathbf{r}} - \Omega\bar{t})]\right], \tag{25}$$

displaying the local propagation vector $\nabla\gamma$ and instantaneous frequency $-\partial_t\gamma$. It must be borne in mind that a "local wave phase" $\nabla\gamma\cdot\mathbf{r} - (\partial_t\gamma)t$ cannot be constructed from (25); therefore, the integration of the line integral must be performed in order to derive ψ. However, (25) clearly displays the first order in v/c correction terms $h_0K/(\Omega k_0)$ and h_0/ω_0. Care must be taken in choosing the parameters for the problem at hand, such that $v/c \ll 1$; otherwise, the above formalism is not adequate.

The velocity (21) is the derivative of the displacement $(\mathbf{v}_0/\Omega)e^{i\mathbf{K}\cdot\mathbf{r} - i\Omega t}$. Depending on the amplitude v_0/Ω, we can define weak and strong effects. For $v_0/\Omega \ll \lambda_0$, where $k_0 = 2\pi/\lambda_0$, a weak effect is defined. This corresponds to $(\omega_0/\Omega)(v_0/c) \ll 1$. For amplitudes $v_0/\Omega \gg \lambda_0$, a strong effect is produced. Examples of such cases will be considered in Sec. III.

Another interesting case is obtained by assuming h to be a spherical wave

$$h = h_0 e^{iKr - i\Omega t}/r\,, \tag{26}$$

where $h_0 =$ const. According to (20) we now have

$$i2\mathbf{K}_0\cdot\hat{\mathbf{r}}(iK - 1/r)h - hK^2 + 2\omega_0\Omega h/c^2$$
$$- 2\omega_0\Omega\mathbf{v}\cdot\mathbf{k}_0/c^2 = 0\,. \tag{27}$$

Here we distinguish two limiting cases: For distances $r \ll C/\Omega$, i.e., $\mathbf{r} \ll \Lambda$, where Λ is the wavelength of the bulk velocity, we have $\nabla h = -\hat{\mathbf{r}}h/r$. For distances $r \gg C/\Omega$, we have $\nabla h = -\hat{\mathbf{r}}iKh$. For the first case we can assume in (27) the limit $K \to 0$; hence, we obtain

$$\mathbf{v} = \hat{\mathbf{r}}v_0 e^{-i\Omega t}/r^2, \quad h = h_0 e^{-i\Omega t}/r\,,$$
$$\mathbf{g} = i\nabla h/\Omega = -i\hat{\mathbf{r}}h_0 e^{-i\Omega t}/(r^2\Omega), \quad h_0 = iv_0k_0\Omega/c\,. \tag{28}$$

Choosing the real part, say, (28) becomes

$$\mathbf{v} = \hat{\mathbf{r}}(v_0/r^2)\cos\Omega t\,,$$
$$h = v_0k_0(\Omega/cr)\sin\Omega t\,, \tag{29}$$
$$\mathbf{g} = -\hat{\mathbf{r}}(v_0k_0/cr^2)\cos\Omega t\,.$$

Note that, for $\Omega \to 0$, $\mathbf{v} = \hat{\mathbf{r}}v_0/r^2$, $h = 0$, $\mathbf{g} = \hat{\mathbf{r}}v_0k_0/c$ as expected, because this is the case $\nabla\cdot\mathbf{v} = 0$ discussed above. The second case is very similar to (21), yielding

$$h = (h_0/r)\cos(Kr - \Omega t),$$
$$h_0 = \frac{2\omega_0\Omega v_0\hat{\mathbf{r}}\cdot\mathbf{k}_0/c^2}{2\omega_0\Omega/c^2 - K^2 - 2\mathbf{k}_0\cdot\hat{\mathbf{r}}K}, \tag{30}$$
$$\mathbf{g} = \hat{\mathbf{r}}Kh/\Omega, \quad \mathbf{v} = \hat{\mathbf{r}}(v_0/r)\cos(Kr - \Omega t)\,.$$

The above examples will be used to study scattering problems and the validity of the Doppler effect analysis.

III. SCATTERING IN SPACE AND TIME VARYING SYSTEMS: PLANE SCATTERERS

A. General observations

The formalism developed above is now used to analyze scattering problems involving moving media and/or moving surfaces. Three situations will be examined: (a) the medium is moving and the scatterer is at rest; (b) the medium is at rest and the scatterer is moving; (c) the medium and the scatterer are moving, such that mass continuity is preserved at the scatterer's surface. Cases (a) and (b) do not preserve continuity. Case (b), where the motion of the scatterer is taken into account and the associated motion of the fluid is ignored, is the common model used for Doppler effect analysis in biomedical and industrial applications. It is shown, for the simple cases discussed below, that for all three cases very similar results are obtained at small distances from the surface. For finite (two-and three-dimensional) objects, it is shown that at large distances the effect of the moving fluid [case (c)] vanishes, and case (b) is obtained as a limiting case, thus vindicating the simpler model (b) widely used for various applications. Of course, even a case (c) analysis is only an approximation as long as factors such as compressibility and viscosity are ignored. It is hoped that the present analysis will contribute to resolving the controversial issues raised previously, concerning the validity of the Doppler effect analysis.

B. Plane scatterer, normal incidence, case (a)

Consider

$$\psi_0 = e^{ik_0x} \tag{31}$$

associated with a time factor $e^{-i\omega_0 t}$. The medium moves along the x axis according to

$$\mathbf{v} = \hat{\mathbf{x}} v_0 \cos(Kx + \Omega t), \quad K = \Omega/c ; \tag{32}$$

i.e., we have chosen a velocity wave propagating in the $-\hat{\mathbf{x}}$ direction. For the sake of simplicity, let us ignore the dispersion and assume $C = c$. From (23) it follows, for $\mathbf{K} = \hat{\mathbf{x}}K$, $\mathbf{k}_0 = \hat{\mathbf{x}}k_0$, that

$$\begin{aligned} h &= h_0 \cos(Kx + \Omega t) , \\ h_0 &= 2k_0^2 v_0/(4k_0 - K) , \\ \mathbf{g} &= -\hat{\mathbf{x}} h/c . \end{aligned} \tag{33}$$

We now wish to compute β according to (10), and since the path can be arbitrarily chosen, we consider

$$\begin{aligned} \beta &= \int_{0,0}^{x,t} (\mathbf{g}\cdot\hat{\mathbf{x}} - d\bar{x}) = -\frac{h_0}{c} \int_{0,0}^{x,0} d\bar{x} \cos(K\bar{x} + \Omega\bar{t}) , \\ &- h_0 \int_{x,0}^{x,t} d\bar{t} \cos(K\bar{x} + \Omega\bar{t}) = -\frac{h_0}{\Omega} \sin(Kx + \Omega t) . \end{aligned} \tag{34}$$

For the weak effects as defined above, h_0/Ω is small, and ψ can be approximated in the form

$$\begin{aligned} \psi &= e^{ik_0x - i\omega_0t + i\beta} \\ &= e^{ik_0x - i\omega_0t} + (h_0/2\Omega)\{\exp[i(k_0 - K)x - i(\omega_0 - \Omega)t] \\ &\quad - \exp[i(k_0 + K)x - i(\omega_0 + \Omega)t]\} . \end{aligned} \tag{35}$$

The representation (35) clearly displays the production of two sidebands at frequencies $\omega_0 \pm \Omega$. On the other hand, for arbitrary h_0/Ω we can represent (34) in terms of a Bessel functions series,

$$\psi = \sum_{n=-\infty}^{\infty} J_n\left(\frac{h_0}{\Omega}\right) \exp[i(k_0 - nK)x - i(\omega_0 + n\Omega)t] . \tag{36}$$

For small arguments $h_0/\Omega \ll 1$, the Bessel functions can be expanded near the origin, $J_n(h_0/\Omega) = (1/n!)(h_0/2\Omega)^n$. If we retain $n = 0, \pm 1$ only, (35) is obtained. For $h_0/\Omega \gg 1$ we can use

$$J_n(h_0/\Omega) \approx (2\Omega/\pi h_0)^{1/2} \cos[h_0/\Omega - \pi(2n + 1)/4] .$$

This shows that all sidebands are equally affected and energy is distributed among the spectral components, such that the amplitudes of the individual sidebands decrease. Subsequently only weak effects will be considered, providing simple examples that can be discussed analytically. The choice of the direction of propagation of $\mathbf{v}$ (32) relative to $\mathbf{k}_0$ is important. Thus, for $\mathbf{v} = \hat{\mathbf{x}} v_0 \cos(Kx - \Omega t)$, i.e., $\mathbf{K} = \hat{\mathbf{x}}K$, together with ψ_0 as in (31), i.e., $\mathbf{k}_0 = \hat{\mathbf{x}}k_0$, we obtain from (23) $h_0 = -(2\omega_0^2/\Omega)(v_0/c)$; hence, for weak effects we need $(\omega_0/\Omega)^2(v_0/c) \ll 1$. Similarly, for $\mathbf{v}$ (32) and $\psi_0 = e^{-ik_0x}$, (23) prescribes $(2\omega_0^2/\Omega)(v_0/c)$.

For case (a) discussed now, the scatterer is a plane located at $x = 0$. Boundary conditions may vary, depending on the materials involved. For simplicity let us assume

$$\psi + \psi_r = 0|_{x=0} , \tag{37}$$

where ψ_r denotes the scattered wave, and since (37) is arbitrarily chosen, there is no need to qualify whether ψ denotes acoustical velocity potential pressure or displacement.

At the boundary $\psi = \exp[-i\omega_0t - i(h_0/\Omega)\sin \Omega t]$; hence, at the boundary we must have

$$\psi_r = -\exp[-i\omega_0t - i(h_0/\Omega)\sin \Omega t] . \tag{38}$$

The solution ψ_r must satisfy the wave equation (9) and the boundary condition (37). Consider

$$\begin{aligned} \psi_r &= -\exp[-ik_0x - i\omega_0t - i(h_{0r}/\Omega)\sin(Kx + \Omega t)] \\ &\quad + A_+ e^{-ik_+x - i\omega_+t} + A_- e^{-ik_-x - i\omega_-t}, \end{aligned} \tag{39}$$

$h_{0r} = (2\omega_0^2/\Omega)(v_0/c)$, $\omega_\pm = \omega_0 \pm \Omega$, $k_\pm = k_0 \pm K$.

Since, as shown below, $A_\pm$ are of first order, the velocity effect in the exponent of the sidebands can be neglected. Furthermore, let $A_+ = -A_-$, and expand (38), (39) to the first order. This yields

$$A_+ = -A_- = \frac{(h_{0r} - h_0)}{2\Omega} = \frac{2\omega_0^2}{\Omega^2} \frac{v_0}{c} \frac{2\omega_0 - \Omega}{4\omega_0 - \Omega} . \tag{40}$$

However, if the effect is not weak, as considered above, additional powers in the expansions will be significant; this will increase the number of sidebands of frequencies $\omega_0 \pm n\Omega$, $n = 1,2,\ldots$ having significant amplitudes.

C. Plane scatterer, normal incidence, case (b)

Now it will be asumed that the medium is at rest and the scattering boundary is moving. This configuration violates the mass continuity of the medium at the surface, and therefore is the model sometimes criticized as being unrealistic. The model can be considered somewhat more plausible for cases where the scatterer is perforated, with holes small in comparison to the wavelength, such that the scattering is not affected, and still the medium can flow through the scatterer. Another way of looking at this dilemma is to concede that the model is unrealistic, but to note that the general practice in many situations is to ignore the fluid motion. Since in biomedical and industrial applications this heuristic approach yields good results, it is concluded that it is worthwhile to proceed and analyze this case.

The incident wave is chosen as

$$\psi = e^{ik_0x - i\omega_0t} . \tag{41}$$

The displacement of the boundary is given by

$$x = x_0 \sin \Omega t, \tag{42}$$

corresponding to a velocity $v = v_0 \cos \Omega t$, $v_0 = x_0\Omega$. We have to find a reflected wave ψ_r such that it satisfies the simple wave equation (2) (without primes), and also satisfies the boundary condition

$$\psi + \psi_r = 0|_{x = x_0 \sin \Omega t} . \tag{43}$$

Let the reflected wave be

$$\begin{aligned} \psi_r &= -e^{-ik_0x - i\omega_0t} + A_+ e^{-ik_+x - i\omega_+t} \\ &\quad + A_- e^{-ik_-x - i\omega_-t} , \\ \omega_\pm &= \omega_0 \pm \Omega, \quad k_\pm = k_0 \pm K . \end{aligned} \tag{44}$$

Observe that $k_0x_0 = (\omega_0/\Omega)(v_0/c)$ is small, according to the weak effect defined above. Also, as for case (a) above, let $A_+ = -A_-$ and these amplitudes will be shown to be of first order. Hence to the first-order approximation (43) prescribes

$$A_+ = -A_- = k_0x_0 . \tag{45}$$

D. Plane scatterer, normal incidence, case (c)

At this point the combined motion of the medium and the scatterer will be taken into account, such that, at the scatterer, medium and interface move simultaneously and mass continuity is preserved. The incident wave ψ is given by (35), and the reflected wave is given by (39). The boundary condition is stated in (43). At the boundary (35) becomes

$$\exp[ik_0x_0 \sin \Omega t - i\omega_0 t - i(h_0/\Omega)\sin(Kx_0 \sin \Omega t + \Omega t)]$$
$$\approx e^{iB\sin\Omega t - i\omega_0 t} \approx e^{-i\omega_0 t}(1 + iB \sin \Omega t),$$
$$B = k_0x_0 - \frac{h_0}{\Omega} = \frac{\omega_0}{\Omega}\frac{v_0}{c}\left(1 - \frac{2\omega_0}{4\omega_0 - \Omega}\right). \quad (46)$$

At the boundary, (39) yields to the first order

$$\psi_r = -e^{-iB_r\sin\Omega t - i\omega_0 t} - 2iA_+e^{-i\omega_0 t}\sin \Omega t$$
$$\approx e^{-i\omega_0 t}[-1 + i(B_r - 2A_+)\sin \Omega t] \quad (47)$$
$$B_r = k_0x_0 + \frac{h_{0r}}{\Omega} = \frac{\omega_0}{\Omega}\frac{v_0}{c}\left(1 + \frac{2\omega_0}{\Omega}\right).$$

Hence

$$A_+ = \frac{1}{2}(B + B_r) = \frac{\omega_0}{\Omega}\frac{v_0}{c}\left(1 + 2\frac{\omega_0}{\Omega}\frac{2\omega_0 - \Omega}{4\omega_0 - \Omega}\right). \quad (48)$$

Comparing (48) to (40), it is seen that for $\Omega \ll \omega_0$ the results coincide. Hence the motion of the boundary will have a negligible effect under these circumstances. On the other hand, for case (b) we have (45) which is different in amplitude. The normal incidence problem is somewhat unusual because of the cancellation occurring in the denominator of h_0 (23) when computing h_{0r}, i.e., when $\mathbf{k}_0 \cdot \mathbf{K} = k_0K$. This effect will not be present in the oblique incidence case discussed below. The least we can say is that the common feature of all three cases (a), (b), (c) is the production of the sidebands $\omega_0 \pm \Omega$, and if only frequencies are measured (as is often the case), then the three mechanisms cannot be distinguished. We can look at the problem in a different way: If in (39), (44) the first-order effects are brought into the exponent, then we get

$$\psi_r = -\exp[-ik_0x - i\omega_0 t - iE\sin(Kx + \Omega t)],$$
$$E_a = \frac{h_0}{\Omega} = \frac{1}{\Omega}\frac{2\omega_0^2 v_0/c}{4\omega_0 - \Omega},$$
$$E_b = -2k_0x_0 = -2(\omega_0/\Omega)(v_0/c), \quad (49)$$
$$E_c = -2k_0x_0 + \frac{h_0}{\Omega} = -\frac{2\omega_0}{\Omega}\frac{v_0}{c}\frac{3\omega_0 - \Omega}{4\omega_0 - \Omega},$$

for cases (a), (b), (c), respectively. This means that if the time-dependent frequency of the reflected wave is monitored, all cases are almost identical for all practical purposes. In particular it will be difficult to distinguish case (b) from (c).

E. Plane scatterer, oblique incidence, case (a)

For this case the medium's velocity to be considered is given in (32). For the incident wave we take $\psi_0 = e^{i\mathbf{k}_0\cdot\mathbf{r}}$, where $\mathbf{k}_0 = \hat{\mathbf{x}}k_{0x} + \hat{\mathbf{y}}k_{0y}$; for simplicity let us assume that the wave vector has no component in the $\hat{\mathbf{z}}$ direction. According to (23)

$$h_0 = \frac{2\omega_0 k_{0x} v_0/c}{2k_0 + 2k_{0x} - K'}, \quad k_0 = \frac{\omega_0}{c}. \quad (50)$$

Subject to (34), the incident wave is given by

$$\psi = \exp[i\mathbf{k}_0\cdot\mathbf{r} - i\omega_0 t - i(h_0/\Omega)\sin(Kx + \Omega t)]. \quad (51)$$

The reflected wave is given by

$$\psi_r = -\exp[i\mathbf{k}_{0r}\cdot\mathbf{r} - i\omega_0 t - i(h_{0r}/\Omega)\sin(Kx + \Omega t)]$$
$$+ A_+e^{i\mathbf{k}_{r+}\cdot\mathbf{r} - i\omega_+ t} + A_-e^{i\mathbf{k}_{r-}\cdot\mathbf{r} - i\omega_- t},$$
$$\mathbf{k}_{0r} = -\hat{\mathbf{x}}k_{0x} + \hat{\mathbf{y}}k_{0y}, \quad k_{0r} = k_0 = \omega_0/c,$$
$$h_{0r} = -(2\omega_0 k_{0x}v_0/c)/(2k_0 - 2k_{0x} - K), \quad (52)$$
$$\mathbf{k}_{r\pm} = \mathbf{k}_{0r} \pm \hat{\mathbf{x}}K, \quad \omega_\pm = \omega_0 \pm \Omega,$$

and we assume that $A_- = -A_+$. It is noted that, strictly speaking, the sidebands in (52) are not satisfying the wave equation (9). However, since $A_\pm$ are of first order, the sidebands have to satisfy only the simple wave equation of media at rest. Furthermore, note that $\mathbf{k}_{r\pm}$ are not codirectional with $\mathbf{k}_{0r}$. This is a manifestation of the so-called abberation effect. This effect is usually emphasized in discussions concerning relativistic electrodynamics,[17] but it is a first-order velocity effect that must be considered in acoustics as well. At the boundary, to the first order, (51), (52) satisfy (37), yielding

$$A_+ = (h_{0r} + h_0)/(2\Omega)$$
$$= -\frac{8\omega_0 k_{0x}^2 v_0/c}{(2k_0 - 2k_{0x} - K)(2k_0 + 2k_{0x} - K)}. \quad (53)$$

F. Plane scatterer, oblique incidence, case (b)

This problem has been analyzed in the electromagnetic context by Van Bladel,[17] and De Zutter.[18] The incident wave is

$$\psi = e^{i\mathbf{k}_0\cdot\mathbf{r} - i\omega_0 t}. \quad (54)$$

The boundary condition is given by (43). The scattered wave is postulated to be

$$\psi_r = -e^{i\mathbf{k}_{0r}\cdot\mathbf{r} - i\omega_0 t} + A_+e^{i\mathbf{k}_{r+}\cdot\mathbf{r} - i\omega_+ t} + A_-e^{i\mathbf{k}_{r-}\cdot\mathbf{r} - i\omega_- t},$$
$$\omega_\pm = \omega_0 \pm \Omega, \quad \mathbf{k}_{r\pm} = \mathbf{k}_{0r} + \mathbf{K}_\pm, \quad (55)$$

where $\mathbf{k}_{0r}$ is given in (52) and the wave equation for media at rest prescribes $|\mathbf{k}_\pm| = \omega_\pm/c$. For a choice $\mathbf{K}_\pm \cdot \hat{\mathbf{y}} = 0$, we therefore have a condition on $K_{x\pm}$,

$$(k_{0x} + K_{x\pm})^2 + k_{0y}^2 = (\omega \pm \Omega)^2/c^2. \quad (56)$$

Again (55) displays abberation, as in (52). Since $A_+ = -A_-$ are first-order terms, the values of $k_{x\pm}$ are irrelevant for satisfying the boundary condition (43). Thus at the boundary we have

$$\psi = \exp(ik_{0x}x_0 \sin \Omega t + ik_{0y}y - i\omega_0 t),$$
$$\psi_r = -\exp(-ik_{0x}x_0 \sin \Omega t + ik_{0y}y - i\omega_0 t) \quad (57)$$
$$\times(1 + i2A_+ \sin \Omega t)$$
$$\approx -\exp(-ik_{0x}x_0 \sin \Omega t + ik_{0y}y - i\omega_0 t + i2A_+ \sin \Omega t).$$

Hence

$$A_+ = -A_- = k_{0x}x_0 = k_{0x}v_0/\Omega. \quad (58)$$

Note that (53), (58) are different but of same order of magnitude.

G. Plane scatterer, oblique incidence, case (c)

Here once again we consider (32) to describe the medium's velocity. The boundary condition is given by (43). The incident wave is given by (50), (51), and the scattered wave is given by (52). At the boundary, to the first order, the incident wave becomes [cf. (46)]

$$\psi = \exp(ik_{0y}y - i\omega_0 t + iB\sin\Omega t)$$
$$\approx e^{ik_{0y}y - i\omega_0 t}(1 + iB\sin\Omega t),$$
$$B = k_{0x}x_0 - h_0/\Omega. \tag{59}$$

The scattered wave becomes

$$\psi_r = -\exp(ik_{0y} - i\omega_0 t - iB_r\sin\Omega t)$$
$$- e^{ik_{0y}y - i\omega_0 t} i2A_+ \sin\Omega t$$
$$\approx e^{ik_{0y}y - i\omega_0 t}[-1 + i(B_r - 2A_+)\sin\Omega t],$$
$$A_+ = -A_-, \quad B_r = k_{0x}x_0 + h_{0r}/\Omega, \tag{60}$$

which should be compared to (47). Similarly to (48) we have here

$$A_+ = \tfrac{1}{2}(B + B_r)$$
$$= k_{0x}v_0/\Omega$$
$$\times\left(1 - k_0\frac{4k_0 - 2K}{(2k_0 + 2k_{0x} - K)(2k_0 - 2k_{0x} - K)}\right), \tag{61}$$

where (61) is the analog of (48).

With this we have finished the analysis for plane scatterers. The general conclusion, as mentioned above, is that for weak effects all three cases give rise to sidebands at frequencies $\omega_0 \pm \Omega$. In particular, a comparison of case (b), (58), and case (c), (61), reveals that we are dealing with effects of the same order of magnitude, and it will require a highly sophisticated experiment to distinguish between the two cases.

IV. SCATTERING IN SPACE AND TIME VARYING SYSTEMS: SPHERICAL SURFACES

A. General observations

Plane scatterers and plane waves as presented in the previous sections are appropriate for high-frequency waves whose wavelength is short in comparison with the radius of curvature of the scattering surfaces. In the present section scattering from spherical surfaces is considered, providing examples for three-dimensional configurations. The bulk velocity of the medium, as well as the bulk velocity of the surface, is chosen in the radial direction. This leads to problems that are simple enough for analytical computation. As for the plane scatterer, three cases will be considered in order to highlight the effects of medium and/or surface motion. For the present problem we adopt h as defined in (30), i.e., the bulk velocity at large distances behaves as an outgoing acoustical wave propagating in the radial direction at a frequency Ω. Note however that the definition of h_0 in (30) depends on k_0; hence, for the present case of spherical waves, h_0 must be reconsidered. Furthermore, the present analysis deals with objects small compared to the wavelength Λ of the velocity. This still allows for a wide range of object sizes, in terms of the wavelength λ of the incident wave. The scattered waves will be studied in the farfield. This simplifies the analysis and highlights the motional effects.

Once again it is shown that all the mechanisms corresponding to the cases (a), (b), and (c) yield similar effects of the same order of magnitude. It is also shown that the effect induced by the motion of the medium is negligible at large distances from the scatterer.

B. Scattering by a sphere, case (a)

The incident wave in the absence of medium velocity is given by (31). Presently, because of the velocity effect, it becomes

$$\psi = e^{ik_0 x - i\omega_0 t + i\beta}$$
$$= e^{-i\omega_0 + i\beta}\sum_{n=0}^{\infty} i^n(2n+1)j_n(k_0 r)P_n(\cos\theta),$$
$$x = r\cos\theta, \tag{62}$$

where $\beta = \beta(\mathbf{r},t)$ has to be defined, r,θ are the radial distance and polar angle, respectively, in a spherical coordinate system, and j_n, P_n denote the nonsingular spherical Bessel functions and Legendre polynomials, respectively. To derive the scattered wave we consider distances $r \ll \lambda$ as in (28). This means that we assume $r = a$, the radius of the scatterer, to be small compared to the wavelength of the bulk velocity. In the absence of medium motion, the scattered wave is given as a solution of the simple (Helmholtz) wave equation in spherical coordinates

$$\psi_{0r} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n(2n+1)\frac{(n-m)!}{(n+m)!}a_{nm}h_n(k_0 r)Y_n^m(\theta,\phi),$$
$$Y_n^m(\theta,\phi) = P_n^m(\cos\theta)e^{im\phi},$$
$$P_n^{-m} = (-1)^m(n-m)!/(n+m)!P_n^m, \tag{63}$$

where a_{nm} are constant coefficients, h_n are spherical Hankel functions of the first kind, and Y_n^m are spherical harmonics involving the associated Legendre polynomials P_n^m. Inasmuch as (63) is a solution of a linear wave equation, it may be recast as an integral (superposition) of plane waves. Explicit expressions are given by Stratton,[19] for example. Note that, for $r \ll \Lambda$, (28), (29) yield h, g that are independent of $\mathbf{k}_0$; hence, the scattered wave is simply given by

$$\psi_r = \psi_{0r}e^{-i\omega_0 t + i\beta}, \quad r \ll \Lambda. \tag{64}$$

This means that for a boundary at rest and $r \ll \Lambda$, the scattering coefficients of (63) apply also to the case where the medium is in motion. For the boundary condition,

$$\psi + \psi_r = 0|_{r=a}, \tag{65}$$

whether the medium moves or is at rest, and (62)–(65) prescribe

$$a_{nm} = 0, \quad m \neq 0, \quad a_{n0} = -j_n(k_0 a)/h_n(k_0 a). \tag{66}$$

In order to study the effects produced by the moving medium, consider the solution at large distances $r \gg \Lambda$, where also $r \gg \lambda$. In the farfield[19] $i^n h_n(k_0 r) \approx e^{ik_0 r}/ik_0 r$; hence, (64) becomes

$$\psi_r = (e^{ik_0 r - i\omega_0 t + i\beta}/ik_0 r)G(k_0 a,\theta,\phi), \quad r \gg \Lambda, \tag{67}$$

where G, the scattering amplitude, depends on $k_0 a$ and directions only. Writing $k_0 r = \mathbf{k}_0 \cdot \mathbf{r}$ in the exponent (67), the scattered wave can be interpreted as propagating in direction $\hat{\mathbf{r}}$. Hence for $r \gg \Lambda$ we now have (30) with $\mathbf{k}_0 = k_0 \hat{\mathbf{r}}$. By integrating with respect to $\mathbf{r},t$, for $r \gg \Lambda$ we obtain

$$\beta = h_0 \left\{ \int_{\infty,0}^{r,0} + \int_{r,0}^{r,t} \right\} \left(\frac{d\bar{r}}{\bar{r}} \frac{K}{\Omega} \cos(K\bar{r} - \Omega\bar{t}) - \frac{d\bar{t}}{\bar{r}} \cos(K\bar{r} - \Omega\bar{t}) \right) \approx \frac{h_0}{r\Omega} \sin(Kr - \Omega t),$$

$$h_0 = -\frac{2\omega_0^2}{\Omega} \frac{v_0}{c}, \tag{68}$$

where the reference point for the phase, i.e., the initial point of integration, is taken at $r \to \infty, t = 0$. In performing the space integration in (68), it is noted that for large r, the change of $1/r - 1/(r + \Lambda)$ over a wavelength is approximately Λ/r^2; i.e., the relative change is Λ/r which is considered small, and therefore $1/r$ can be treated as a constant. Comparing (34) and (68) it is noted that $\beta \propto 1/r$; i.e., the motional effect decreases as the distance of the observer from the scatterer increases. This shows that the medium motion is not the paramount factor in producing frequency shifts in the farfield. This conclusion will be supported by the analysis of the subsequent cases.

C. Scattering by a sphere, case (b)

This situation of a medium at rest and a moving surface has been treated before,[3,4] and has been criticized by Piquette and Van Buren[1] and previously by Rogers.[5] The incident wave is given by (62) with $\beta = 0$. The boundary condition is prescribed as

$$\psi + \psi_r = 0|_{r = a + \xi \sin \Omega t}\,. \tag{69}$$

At the boundary ψ becomes

$$\psi = e^{-i\omega_0 t} \sum_{n=0}^{\infty} i^n (2n+1) j_n (k_0 a + k_0 \xi \sin \Omega t) P_n(\cos\theta)\,. \tag{70}$$

The displacement $\xi \sin \Omega t$ corresponds to a velocity $v_0 \cos \Omega t$, where $\xi = v_0/\Omega$. For $v_0/(\Omega a) \ll 1$ in (70), we have a weak effect for which only first-order terms should be retained. Expanding and keeping only these terms yield

$$j_n(k_0 a + k_0 \xi \sin \Omega t) \approx j_n(k_0 a) + j_n'(k_0 a) k_0 \xi \sin \Omega t, \tag{71}$$

where j_n' denotes differentiation with respect to the argument. Inserting (71) into (70) displays the production of sidebands $\omega_0 \pm \Omega$. It is therefore necessary, in order to satisfy the boundary conditions, to postulate a scattered wave with spectral components ω_0, $\omega_0 \pm \Omega$. At the same time the scattered wave must have the structure prescribed by (63), in order that the wave equation be satisfied. Consider

$$\psi_r = \sum_{n=0}^{\infty} i^n (2n+1) P_n(\cos\theta) [a_{0n} h_n(k_0 r) e^{-i\omega_0 t} + a_{n+} h_n(k_+ r) e^{-i\omega_+ t} + a_{n-} h_n(k_- r) e^{-i\omega_- t}],$$
$$\omega_\pm = \omega_0 \pm \Omega, \quad k_\pm = \omega_\pm / c, \tag{72}$$

as a candidate for the scattered wave, and assume that $a_{n\pm}$ are already of the first order (this assumption is verified below). Obviously (72) is a proper solution of the wave equation. At the boundary, and keeping only first-order terms, we obtain the condition

$$j_n(k_0 a) + j_n'(k_0 a) k_0 \xi \sin \Omega t + a_{0n} h_n(k_0 a) + a_{0n} h_n'(k_0 a) k_0 \xi \sin \Omega t + a_{n+} h_n(k_+ a) e^{-i\Omega t} + a_{n-} h_n(k_- a) e^{i\Omega t} = 0\,. \tag{73}$$

Due to the orthogonality of the time function in (73) this yields

$$a_{0n} = -j_n(k_0 a)/h_n(k_0 a)\,,$$
$$a_{n+} h_n(k_+ a) = -a_{n-} h_n(k_- a)\,,$$
$$k_0 \xi [j_n'(k_0 a) + a_{0n} h_n'(k_0 a)] = 2i a_{n+} h_n(k_+ a)\,, \tag{74}$$

which is a solution for $a_{n\pm}$, of order $k_0 \xi = (\omega_0/\Omega)(v_0/c)$, i.e., a weak effect by definition [provided the term in brackets in (74) is nonsingular, which is the case for all real k_0 because poles of $h_n(k_0 a)$ are complex]. It is interesting to note that all spectral components in (72) have the same dependence on distance. Although the amplitudes $a_{n\pm}$ are small compared to a_{n0}, once the sidebands are created they fall off with distance according to $1/r$ like the spectral component at the center frequency ω_0. It follows that the Doppler effects produced by the moving boundary are *not* decreasing as we move away from the surface, as opposed to the effects produced in case (a) above.

D. Scattering by a sphere, case (c)

Finally, we compute the combined effect of moving medium and boundary. This effect is in fact a juxtaposition of the previous cases (a), (b), and could be handled simply by inspection of the previous results, but because it is basic to the discussion, it will be derived in detail. The incident wave is now (62), where β is obtained for the range of distances close to the scatterer $r \ll \Lambda$ as

$$\beta = \frac{v_0 k_0}{c} \left\{ \int_{\infty,0}^{r,0} + \int_{r,0}^{r,t} \right\} \left(-\frac{d\bar{r}}{\bar{r}^2} \cos \Omega\bar{t} - \frac{\Omega\, d\bar{t}}{\bar{r}} \sin \Omega\bar{t} \right)$$
$$= (v_0 k_0 / cr) \sin \Omega t, \tag{75}$$

where g, h are given by (29). The velocity effect caused by the moving medium has to be included in the scattered wave, as given in (64). However, in the sidebands in (72), which are already correction terms, the medium's velocity effect can be neglected. The effect in (73) will be a factor $e^{i\beta}$ multiplying $j_n(ka)$ but no other term. Consequently, the value of $a_{n\pm}$ is not affected. In the farfield β is computed as in (68). This simply means that $a_{n+} h_n(k_+ a)$ must be replaced now by

$$a_{n+} h_n(k_+ a) + h_0/(r\Omega)\,. \tag{76}$$

As in case (a), the effect of the moving medium decreases as the distance r increases, but the effect produced by the moving boundary is independent of the distance.

V. SUMMARY AND CONCLUDING REMARKS

The present study was motivated by an ongoing controversy, dating back to Doppler's original work.[7,8] For some cases it has been argued that because the medium surrounding the scatterer is also moving, in order to preserve mass continuity, the Doppler effect cannot take place. Even if this is not so, i.e., the Doppler effect exists, to neglect the effect of the moving medium is arbitrary if we do not know the significance of this effect. The theoretical analysis of such problems is very complicated, and for practical purposes, e.g., biomedical and industrial applications, the medium motion is usually neglected. Experience has taught experimentalists that this approach yields reasonable results within the expected accuracy of the systems used, even though an appropriate explanation was not available. Using a specialized and restricted model, the present study tries to provide an analytical basis for the physical situation, and to examine to what extent the previous heuristic approach is justified.

The wave equation for moving media is derived subject to a Galilean transformation (essentially incorporating the material derivative). A solution is derived for this equation. This solution is a ray-type (WKB) correction to the velocity-independent solution. Medium effects are studied and scattering problems are analyzed. Three cases are examined: (a) the medium moves and the scatterer is at rest; (b) the medium is at rest and the scatterer moves; (c) both scatterer and medium are in motion. Plane scatterers are considered for normal and oblique incidence. Scattering by spheres is considered to provide examples for bounded scatterers.

The conclusion of the discussion is that the effects of the moving medium are of the same order of magnitude as those produced by the moving surfaces. Furthermore, in realistic three-dimensional cases one should expect the medium effects to vanish in the farfield. The Doppler effect analysis, based on a linear wave equation and time-dependent boundary conditions, seems to be an adequate tool for practical purposes, at least for the weak effects discussed in the above examples. In situations where the medium effects are too complicated for analysis, it is still worthwhile to consider the motion of the boundaries, even though such a model violates mass continuity. Engineers and experimentalists have been doing this now for decades and found their results to be of satisfactory accuracy.

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