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SIMULATION OF LIGHT BEAM PROPAGATION IN NONLINEAR MEDIA

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ABSTRACT

The present paper concerns the simulation of nonlinear wave propagation in the ray regime, i.e., in the limit of geometrical optics. The medium involved is nonlinear. Linear ray propagation is conventionally computed by using Hamilton's ray equations, whose inhomogeneous terms are derived from the dispersion equation. The formalism used to solve such a set of equations is the Runge-Kutta algorithm. In the present case of nonlinear propagation, the inhomogeneous terms depend on field amplitudes which are heuristically determined by the convergence (or divergence) of the rays in the beam. However, in the present case the varying convergence depends on the solution of the Hamilton equations, and it is therefore necessary to modify the original Runge-Kutta scheme, by building into it some iteration mechanism, such that the process converges to values which take into account the amplitude effect. As expected the results display self-focusing effects characteristic of nonlinear optics problems.

1. INTRODUCTION

1.1 Dispersion Equation

Let us assume that the media is changing slowly in space and time, compared to wavelength and period of the oscillations. We assume that, locally and instantaneously, we have plane waves [1], written in the form

$$f(\mathbf{r},t) = \mathrm{I}(\mathbf{r},t)e^{i\theta(\mathbf{r},t)} \tag{1}$$

where I is the field amplitude, **r** the space coordinates, t time and θ phase function of a plane wave. The existence of the Eikonal approximation (1) is assumed. Substituting (1) in Maxwell equations, and allowing the constitutive relations to change slowly in space and time, we obtain the dispersion equation

$$F(\mathbf{k},\omega,\mathbf{r},t) = 0 \tag{2}$$

k is the propagation vector, ω is angular frequency, and **r**, t signify that the constitutive parameters $\overline{\varepsilon}, \overline{\mu}$ can slowly change in space and time. The dispersion equation (2) can be written as a differential equation on θ

$$F(\partial\theta/\partial\mathbf{r}, -\partial\theta/\partial t, \mathbf{r}, t) = 0 \tag{3}$$

In this form it is known as the Eikonal equation.

1.2 Hamilton Equation

Apart from of simple cases, (3) is rather complicated and analytical solutions are not available. Therefore, an equivalent set of coupled first-order differential equations is constructed which is easier to compute numerically. These are the Hamilton ray equations [1]. The ray equation can be derived by using the Fermat principle [1], [2]

$$\frac{d\mathbf{x}}{dt} = -\frac{\partial F/\partial\mathbf{k}}{\partial F/\partial\omega} = \mathbf{v}_g \tag{4a}$$

$$\frac{d\mathbf{k}}{dt} = \frac{\partial F/\partial\mathbf{x}}{\partial F/\partial\omega} \tag{4b}$$

$$\frac{d\omega}{dt} = -\frac{\partial F/\partial t}{\partial F/\partial\omega} \tag{4c}$$

where F is the dispersion equation.
The solution of (4) yields the ray path **x**(t) and on it, the values of **k**(t), ω(t).

1.3 Weakly Nonlinear Media

Nonlinear media for electromagnetic waves are obtained by field amplitudes that affect the effective parameters (e.g., dielectric constant) of the media. In this case, the trajectories of the beam are influenced by the field amplitude. Weakly nonlinear media characterized by constitutive relation of the form [3]

$$D_{qi} = \bar{\varepsilon}^{(1)}_{ij} E_{qj} + \bar{\varepsilon}^{(2)}_{ijk} E_{qj} E_{qk} \ldots + \bar{\varepsilon}^{(n)}_{ij\ldots\nu} E_{qj} \cdots E_{q\nu} + \ldots \quad (5)$$

where i=1, 2, 3 denotes Cartesian components, D_i, E_j, E_k the amplitude of the ith component of the qth harmonic, and the parameter $\bar{\varepsilon}^{(n)}{}_{i,j,\ldots}$, are tensors characterizing the system define in [3]. The last expression (5) describes the field as an infinite sum of terms, starting with the linear then the first nonlinear and continuing to a higher degree of nonlinearity. We assume that higher terms in (5) are decreasing significantly, hence for practical purposes (5) may be truncated. In the frame of this paper only the fundamental harmony and the first two terms in (5) are considered, i.e., the linear and the first nonlinear term. $\bar{\varepsilon}^{(2)}_{ijk} E_j E_k$ describes the dependence of dielectric coefficient on the amplitude and therefore

$$\mathbf{D} = \bar{\varepsilon}^{(1)} \cdot \mathbf{E} + \bar{\varepsilon}^{(2)}_{jk} \cdot \mathbf{E} \cdot \mathbf{E} \quad (6)$$

By using two dimensional axes only, the problem is more simple; thus $\bar{\varepsilon}^{(1)}$ is considered as a scalar constant, and the other conjugate is $\bar{\varepsilon}^{(2)} E^2$. we assume that in spite of the dependence of the dielectric coefficient on the fields amplitude, they are constants, which lead to a new definition of the dielectric coefficient

$$\tilde{\varepsilon} = \bar{\varepsilon}^{(1)} + \bar{\varepsilon}^{(2)} E \quad (7)$$

$\tilde{\varepsilon}$ is the total dielectric coefficient, $\bar{\varepsilon}^{(1)}$is the linear part of the dielectric constant, and $\bar{\varepsilon}^{(2)} E$ is its nonlinear part depending on the field amplitude. In this case the fields **E**, **D** point to y direction (the system coordinates will be explained in the next section).

2. THE MODEL

2.1 Description of The System Axis

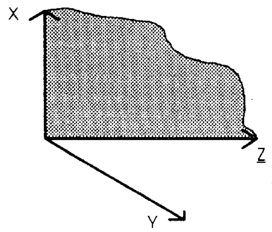

Fig. 1. The system Axis

Only a simple case of two dimensions, x, z as components of the system axis is discussed. Hence, the field **E** coincides with y axis direction, as described in Fig.1. The assumption simplifies the calculations. But we still hope that the characteristics of ray propagation in nonlinear media are retrained. The model is established on the following assumptions: **a.** energy conservation i.e., no energy is added nor dose any leave the beam; therefore in the two-dimension geometry of the problem, the field amplitude depends on the distance between the rays. **b.** the amplitude variance is determined by the convergence (or divergence) of the rays in the beam and effects $\tilde{\varepsilon}$, which is depends on the amplitude. **c.** the rays enter the media, are paraxial with z axis. **d.** the ray on x=0 axis propagate only on z axis direction. **e.** there are no frequency changes, ω=const, therefore from (4c) dω/dt=0

2.2 Characterization of The Problem

Consider the media characterized by dispersion equation

$$F = F(\mathbf{k}, \omega, \mathbf{x}, \mathrm{E_x}) = k_x^2 + k_z^2 - \omega^2 \mu_0 \tilde{\varepsilon} = 0 \quad (8)$$

μ_0 is permeability and E_x is the field amplitude. The medium involved is non magnetic therefore $\mu_r = 1$. The nonlinearity is determined by the dependence of the dielectric coefficient $\tilde{\varepsilon}$ on the amplitudes

$$\tilde{\varepsilon} = \varepsilon_0(\varepsilon_r + \beta E) \tag{9}$$

β is the amplitude influence degree on the media; if β is greater, then the media is more nonlinear.
The Hamilton equations for the specific system (8) are obtained by assigning dispersion equation (8) into Hamilton equation (4) [3]. $\tilde{\varepsilon}$ depends only on x axis. We assume that the changes of $\tilde{\varepsilon}$ in two traced point comparatively to the propagation on z axis can be ignored, therefore explicit sets of differential first degree equations describing the system are obtained

$$\frac{dz}{dt} = \frac{k_z}{\omega \mu_o \tilde{\varepsilon}} \tag{10a}$$

$$\frac{dx}{dt} = \frac{k_x}{\omega \mu_o \tilde{\varepsilon}} \tag{10b}$$

$$\frac{dk_x}{dt} = \frac{\omega\, \partial\tilde{\varepsilon}/\partial x}{2\tilde{\varepsilon}} \tag{10c}$$

$$\frac{dk_z}{dt} = 0 \tag{10d}$$

The set of equation in (10) is solved by the Runge-Kutte Algorithm. Equation (10d) denotes that variance on k_z is also ignored, but (8) is still correct (equal to zero).

2.4 Energy Conservation Law and Poynting vector

Hamilton equations do not include information on the fields amplitude. Since the media depends on the fields amplitude, energy conservation is used to obtain information on the field amplitudes, thus the dielectric permittivity $\tilde{\varepsilon}$ is recalculated according to (9). Some assumption has to be made in order to use the energy conservation: **a.** The wave used is a plane wave (the Hamilton equations are developed by substituting planar waves in Maxwell's equation). **b.** It is assumed that Poyting vector (energy flux density) is valid for the weakly nonlinear media

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \tag{11}$$

The integration of the Poynting vector over a closed surface yields the energy crossing the surface per time unit in an outward sense [5]. In the present case of two dimensions, the energy is calculated by the energy flux density multiplied by the perpendicular distance between the rays.

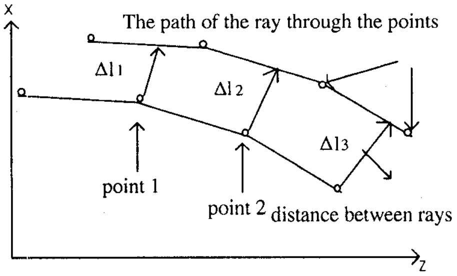

Fig. 2. Energy flux and propagation of the beam

Fig.2 describes the energy flux. Δl1, Δl2 are the distance between the rays. Δl1 is measured first, then the rays propagate to a new location and Δl2 is measured. To maintain the energy conservation, the energy flux trough Δl1, must flow through Δl2, e.g., if the rays converge, then Δl1>Δl2, therefore perpendicular lines to the rays have to be found, in order to find the energy flux passing through them. With the assumption of plane wave in the present case Poynting vector becomes

$$P = E_1{}^2 \sqrt{\frac{\tilde{\varepsilon}_1}{\mu}} \Delta l_1 = E_2{}^2 \sqrt{\frac{\tilde{\varepsilon}_2}{\mu}} \Delta l_2 \tag{12}$$

E1 is the field amplitude on point 1 (Fig.2), E2 is the field amplitude on point 2, Δl1 is the distance between the rays on point 1, Δl2 is the distance between the rays on point 2, μ is equal on both sides thus it vanish. $\tilde{\varepsilon}_1$ is the dielectric permittitvity on point 1, $\tilde{\varepsilon}_2$ is the dielectric permittitvity on point 2. Since weakly nonlinear is assumed, the variation of the dielectric permittitvity $\tilde{\varepsilon}$ compare to the propagation of the rays between two trace points is considered very small, therefore those terms can be reduced on (12), given

$$E_2 = E_1\sqrt{\Delta l_1/\Delta l_2} \qquad (13)$$

In order to improve the results according to (13), an iterative model is used. The simulation uses the value of $\tilde{\varepsilon}$ between two traced points to calculate their mean, then using the new median $\tilde{\varepsilon}$, the simulation returns to the first point and recalculate a new coordinate for the rays. Fig.2 shows the propagation of two rays trough points, the last two points that the rays reach are the basis for the next calculation. By using small time intervals the points are the rays trajectory, and the perpendicular lines describe the wave front.

2.5 Simulation of Loss Media

Losses in the media are applied by multiplying (13) with a loose constant (1-$\gamma\Delta z$)

$$E_2 = E_1\sqrt{\Delta l_1/\Delta l_2}\cdot(1-\gamma\Delta z) \qquad (14)$$

It is well-known that the field amplitude of a plane wave propagating in a loss media will be multiplied by $e^{-\gamma z}$, as it propagates in z axis (if this is an amplification media, e.g., laser then γ is negative and the function will rise exponentially). For small steps of z, $e^{-\gamma\Delta z} \cong 1-\gamma\Delta z$, i.e., the amplitude of a wave propagationate trough distance Δz is linear relative to Δz, with a proportion constant. Therefore, the decision to multiply the fields amplitude with a constant (1-$\gamma\Delta z$) is suitable to the assumption that the ray in loss media is attenuated exponentially due its propagation. Note if γ=0, then this is a nonloss media.

3. RESULTS

3.1 Testing the Simulation

Three cases are described. The first uses uniform amplitude distribution for testing the simulation. The other two describe the response of nonlinear media to major distributions, gaussian and exponential.

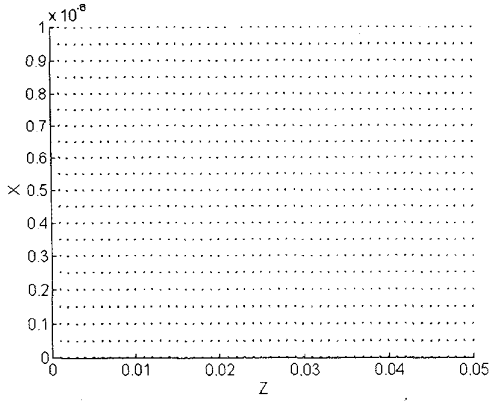

Fig. 3. beam propagation for uniform distribution

In the first case, Fig.3 the amplitude distribution used as initial distribution is uniform. In spite of the dependence of the media on the field amplitudes (β >0), the refractive index is uniform to all the media, therefore the gradient of the dielectric permittivity is zero, hence the rays propagate in paraxial to z axis. The media is characterized by n=1.5, $\beta=10^{-7}$, γ=0, and time interval $5\cdot10^{-12}$ sec.

3.2 Response to Major Distributions

For the other two distributions, three cases for exponential distribution is presented and one for the gaussian distribution. The first Fig.4 uses arbitrary values of n=1.5, $\beta=10^{-8}$ and γ=0 (no losses); the second Fig.5 uses a stronger nonlinearity n=1.5, $\beta=10^{-7}$ (the parameters are very close to "toluene" [6]) and γ=0; the third Fig.6 uses n=1.5, $\beta=10^{-8}$ and (1-$\gamma\Delta z$)=0.95 (losses media). For the gaussian distribution one case only is presented. Fig.7 with parameters values of n=1.5, $\beta=10^{-8}$ and γ=0 (no losses).

For all cases time interval 10^{-12} sec.

4. SUMMARY AND CONCLUSIONS

For a uniform distribution and also for β=0 (not described) the rays logically propagate in paraxial to z axis see Fig.3. As expected the results shows the self-focusing phenomenon.

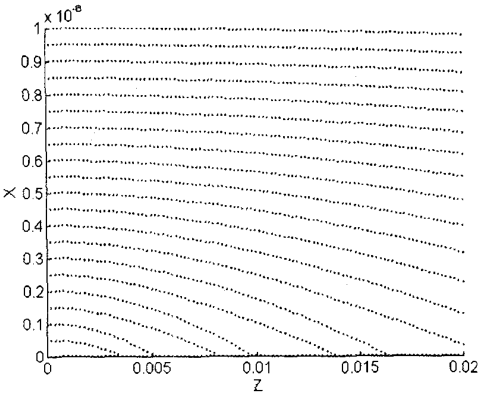

Fig. 4. self-focusing exponential distribution, $\beta = 10^{-8}$

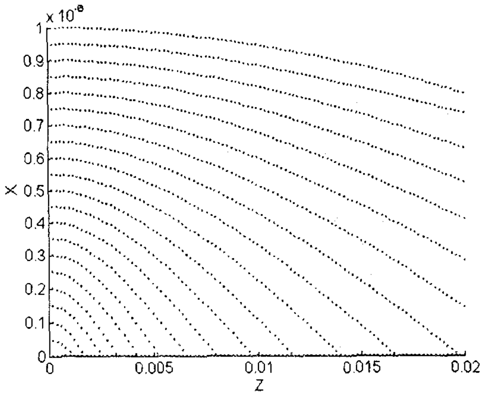

Fig. 5. self-focusing exponential distribution, $\beta = 10^{-7}$

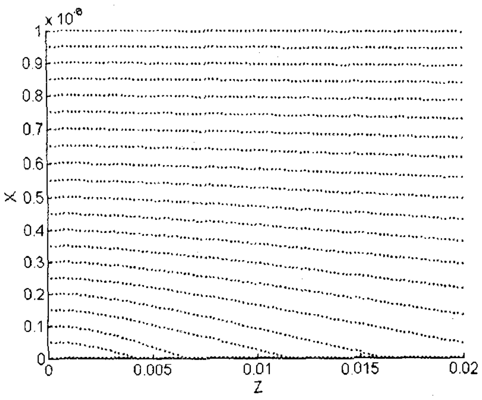

Fig 6. losses in the nonlinear media, $(1-\gamma\Delta z) = 0.95$, $\beta = 10^{-8}$

Likewise, twisted zones of the amplitude distributions profile cause sub self-focusing points Fig.7. Therefore the shape of the initial amplitude distribution influences the form of the self-focusing. β define the dependence of the amplitudes on the media, greater β causes the media to be more nonlinear as shown in Fig.4, Fig.5. γ defines the losses on the media. Fig.6 show that losses media restraint the convergent of the nonlinearity.

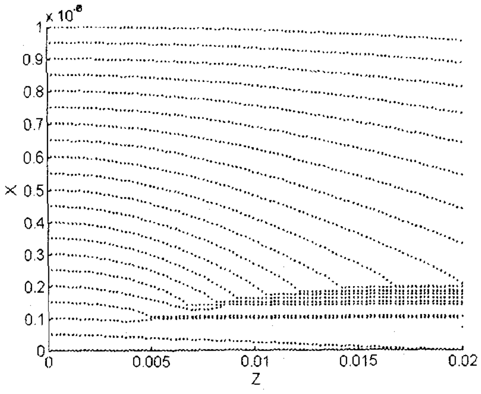

Fig. 7 self-focusing for gaussian beam profile

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