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APPLICATION-ORIENTED RELATIVISTIC ELECTRODYNAMICS

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Summary

The following is a summerized version of the complete text to appear in PIER- Progress In Electromagnetic Research (Editor-in-chief J.A. Kong) intended to accompany the talk in the present conference.

Relativistic electrodynamics is nowadays recognized as pertinent to many practical applications. We need to define a suitable syllabus and explore the best methods for educating future generations of engineers and applied physicists. The attempt presented here is of course biassed by personal preferences. It is argued that relativistic electrodynamics should be presented axiomatically, without trying to "explain the physical meaning" of special relativity, and that four-vectors and their mathematical properties should be emphasized, and that the field tensors, a formalism of limited practical use, should be avoided. Use of four-fold Fourier transforms greatly simplifies the relevant manipulations, and is also important for discussion of dispersive media. This approach yields many concepts as mathematical results, e.g., the relativistic Doppler effect, which therefore do not require a long phenomenological discussion with many "explanations". One of the main results shown here is the fact that the generalized Fermat principle states that the ray will propagate in such a manner that the proper time will be minimized (or extremized, in general). It also strips the mystique of this principle, showing that it is in fact equivalent to a modest mathematical condition on the smoothness of the phase function. The presentation is constructed in a way that allows the student to gradually overcome difficulties in assimilating new concepts and applying them. In that too it is different from many conventional presentations.

Introduction and Rationale

The intimate relationship between Maxwell's theory for the electromagnetic field and Einstein's special relativity theory is generally recognized nowadays. Throughout the present century many educators found it necessary to include a chapter on special relativity in textbooks devoted to electromagnetic field theory. It becomes apparent that a direct approach suitable for educating applied physicists and electromagnetic radiation engineers is lacking. Some authors introduce special relativity theory in the traditional "Gedanken experiment" approach, and by the time the reader finishes with the moving trains, flashing torchlights, and rods and clocks, the relevance to practical electromagnetic problems is obscured. Others move along more formalistic lines and derive the field tensors, mostly by using general coordinate systems and the heavy machinery of differential geometry, i.e., covariant and contravariant coordinate systems, but the mathematical elegance hardly inspires the engineering student who usually cannot find in it any motivation to move on in this field. On the other hand, we are nowadays aware of real life problems, e.g., design of satellite supported global navigation and positioning systems, which involve special (and sometimes even general) relativitstic considerations related to precision of time and frequency bases and errors incurred during propagation through complicated inhomogeneous and time varying media, and everything in the presence of relative motion between objects. It is therefore mandatory to devise the methodological tools and suitable representations for teaching relativistic electrodynamics to applied physics and electrical engineering students. In the course of such a pedagogical experiment with electrical engineering graduates, it became clear that the rudiments of special relativity should be presented axiomatically, with as little phenomenological "explanations" as feasible, working on the assumption that this aspect has been covered at least to some extent in "Baby physics" courses. To repeat this part of the story means that in a one quarter (or semester) course there might not be sufficient time for effectively discussing the more advanced topics presented here. It also became clear that four-dimensional Fourier transforms should be introduced right from the beginning, an unorthodox approach as far as this author is aware. This facilitates the work in an algebraic, rather than differential equations environment, thus simplifying mathematical manipulations. It also became clear that four-vectors, which are easily handled, almost as easily as the classical three-vectors, should be extensively used. Most of the students met had a fair to good grasp of vector analysis and linear algebra, and the introduction of four-vectors and dyadics did not pose a problem. However, only Euclidian systems are considered, and even in this context, the elegance of the field tensors and the associated representation of Maxwell's equations has been avoided. Within these limits, it is then the personal preference of the teacher that will guide him to emphasize certain classes of problems. From this point of view the specific material described here serves merely as an example. Surprisingly, in the course of compiling and teaching this course, new ideas and representations emerged, which do not appear in the literature, as far as this author is aware. These innovative concepts do not alter special relativity theory, still it is a pleasant surprise that at the turn of the century anything new at all can be said about the now veteran special relativity theory. From this point of view there are some novel ideas given here and the present study is not exclusively tutorial. For example, the section on the Fermat principle shows that the generalized principle, for inhomogeneous and time dependent media, acquires a new meaning that can only be stated in the context of special relativity: Verbally stated, it says that the ray propagates along a path that minimizes (or in general extremizes) the proper time. It is also shown that the Fermat principle is equivalent to a simple mathematical condition on the smoothness of the phase function.

The present syllabus is organized as follows: After introducing notation and stipulating relativistic electrodynamics axiomatically, and exploring properties of relevant four-vectors, the technique of algebraization by using four-fold Fourier transforms is introduced. This already touches on the important and nontrivial question of relativistic transformations related to representation space. Further exploration of four-vectors follows. Next, four-potentials are introduced. This is followed by a discussion of the cross multiplication operation and the related rotor operation. It is mentioned, without going into too much detail, that we are dealing with tensor operations, and the general advice is to work with the various components (this is done without mentioning the antisymmetric tensors and their properties, which would encumber the presentation without contributing to application-oriented problems). A section on the proper time and related concepts follows. What might sometimes appear as a melange of unrelated subjects is actually an attempt to lead the student gradually from the less complicated to the more sophisticated subjects. We now have enough tools for discussing specific problems. As an example the Minkowski constitutive equations for moving media are derived for dispersive anisotropic media. Dispersion equations and their relativistic invariance are discussed. This provides the basis for discussing Hamiltonian ray propagation for inhomogeneous and time varying dispersive media. For pedagogical reasons this section is separated from the discussion of the generalized Fermat principle, given in the following section.

Special Relativity

In this section relativistic electrodynamics is introduced. The formalism needed by the applied physicist and engineer is stipulated in an axiomatic manner. The introduction of the field tensors and the ensuing elegant representation of the field equations by means of operations on these tensors, a cornerstone of relativistic formalism, is obviated. Four-vectors and Minkowski space is introduced at the end as a notational and operational tool, rather than a conceptual generalization of the space-time manifold idea, as given in books specializing in relativity theory. In the following sense it is the same methodology educators use now for years when teaching for example waves in metallic waveguides and resonators: The fact is that our students never get a comprehensive course in the theory of the special functions needed for this subject, but we realize that we do not have the time to plough this field, lest no time will remain to teach the pertinent engineering aspects. Therefore a short resume of the theory of Bessel functions, Legendre polynomials, etc. sometimes appears at the end of a textbook for later reference, meanwhile we force our students to plunge into the main subject matter, hoping they will swim and not sink.

Fourier Transforms and the Doppler Effect

In this section the four-dimensional Fourier transform is considered, in terms of the relevant four-vectors. The relativistic Doppler effect is deduced as a simple consequence of the properties of the four-gradient operation. The four-gradient in representation (i.e., **K**-) space is stated. The transformation formulas for the field components in **K**-space are established, this involves the invariance of the four-dimensional volume elements in the various spaces. Once Maxwell's equations and the field transformation formulas are available in algebraic form, it becomes much easier to manipulate the expression, e.g., to show that by substitution of the field transformation formulas into the unprimed set of Maxwell's equation, the primed set is derived.

Invariants Galore

In a sense, all physical laws and models are declarations about the invariance of certain quantities. Conservation laws are obviously in this category, but many other properties, e.g., symmetry in whatever sense, is also a declaration that something is unaffected, or conserved, or invariant, subject to some operation. Even writing a mathematical (algebraic, differential, integral etc.) equation for a physical law, such that everything appears on the left and is equal to zero on the right, is a declaration that "something" (the expression on the left) is immutable, i.e., equal to zero.

The scalar product of two four-vectors is one way of deriving Lorentz invariants, some of them have been recognized as fundamental laws, others are less important, but stand by for whenever they might be used. Another way of deriving invariants is through the field transformation formulas. Of course this is related to the properties of the field tensors, but can be easily verified directly. We mention the volume elements for the various spaces associated with the four-vectors as another way of deriving invariants. Finally it is noted that through the use of the proper time, new four-vectors and associated invariants can be derived.

Potentials

As a variation of the theme, the potentials will be discussed in the context of the Fourier transformed algebraic Maxwell equations. The Maxwell equations are split into sets involving fields excited by electric (conventional) and magnetic sources. The electric and magnetic four-vector potentials are derived and discussed.

The Cross Multiplication and Curl Operations

Teachers of a first course in electromagnetic field theory at sophomore or junior level are aware of the fact that vector analysis, in particular the Curl operation, are a major stumbling block for most students. Witness the long introductory chapters or detailed appendices in most textbooks. Suddenly, after some assimilation of the new concepts took place, they are told in the context of relativistic electrodynamics that the Curl operation is "not really a vector operation", actually an antisymmetric tensor with certain properties. In a short and condensed course it was found expedient to keep tensor analysis and the formal details to the absolutely necessary minimum. It is easy to see that a construct $A_iB_j - A_jB_i$ is an antisymmetric matrix. This in general defines the Curl operation where we now have $A_i = \partial/\partial X_i$. For i,j = 1, 2, 3 there are only three independent entries in the matrix, therefore the Curl operation in three dimensional space could be disguised as a vector operation, on the other hand in four dimensional space i,j = 1, 2, 3, 4 there are six independent entries, therefore there is no way that such an entity could be represented as a four-vector. This discussion is considered sufficient for a first course in applied relativistic electrodynamics.

Proper Time and Related Concepts

In a subsequent section ray equations are considered. The concept of a ray is intimately associated with wave packets and their motion in space. For that and other purposes we have to include a short section on the concept of proper time and related concepts of velocity and acceleration. Actually it is also warranted on ground of intrinsically being an ingenious idea: The creation of new four-vectors by associating four-vectors with invariants, e.g., the proper time, as done below.

The four-velocity and four-acceleration are defined, and their properties and transformations discussed. This is a good time to pick up the subject of the Coulomb and Lorentz force formulas stated in the beginning, and discuss a few general subjects of relativistic mechanics, e.g., the momentum-energy four-vector.

The Minkowski Constitutive Relations

Sommerfeld discusses the Minkowski constitutive relations for moving media. The question is an old one, and can be asked in various ways. If you ask "how does a moving medium behave, for example, does it appear to be a different medium with different constitutive parameters?", then the answer to the question is given in terms of the transformation formulas for the constitutive parameters. This has been amply discussed in the literature, but this author's opinion is that this manner of asking the question does not contribute to any problem of application-oriented relativistic electrodynamics. The question should be asked in the way Minkowski asked it: What are the relations between the fields in a moving medium. Even this definition is not as practical as the direct derivation of the dispersion equations, discussed below. Sommerfeld considers the simple case of a medium which is linear, isotropic nondispersive and homogeneous in its rest frame, i.e., the comoving frame of reference. The treatment is not much more complicated when anisotropic dispersive media are assumed. A bonus of this approach is that we can now mention, through the subject of dispersive systems, the problem of generally non-local and non-instantaneous processes, and its relation to the light cone and causality.

Dispersion Equations in Moving Media

The concept of a dispersion equation is central to wave propagation in general, and especially in connection with ray propagation in dispersive media, discussed subsequently. It is therefore essential for engineers and applied physicists to cover this subject in the course of discussing relativistic electrodynamics. The derivation of the dispersion equation is discussed, and its Lorentz invariance (and the meaning of this phrase) for various inertial observers is established.

Application to Hamiltonian Ray Propagation

The subject of ray propagation in dispersive media is important for applied physicists and electromagnetic radiation engineers. It serves to compute field problems in dispersive inhomogeneous time-varying media, e.g., magnetized plasma problems appearing in connection with ionospheric radio wave propagation. Usually it is presented in the literature as a consequence of the Fermat principle, which is mathematically stated in terms of a variational principle. The subject is presented here in a simplified, although concise manner, which obviates the necessity of introducing the Fermat principle as a variational principle. This was found as a pedagogically preferable approach for the author's students. The Fermat principle (discussed here in the following section), is then presented when the student is already familiar with the Hamilton ray equations and possesses a basis for comparison. Ray propagation also serves here as an example for using four-vectors, for extending the **K** space beyond the Fourier transform, and it clarifies the role of the group velocity in ray theory.

In order to introduce the subject and relevant concepts, we start with the transition from general wave functions to wave packets in homogeneous media. The definition of wave packets in inhomogeneous, time dependent media is impossible in general. However, for "slow variation" such that the variation of the properties of the medium over distances and time intervals commensurate with the wavelength and the period, respectively, of the signal, an approximate procedure can be defined. This is usually referred to as working in the high frequency limit. Clearly spatial and temporal changes in the constitutive parameters do not fit into our formalism for homogeneous media because they are not consistent with a Fourier transform representation, nor is a representation of a wave function in terms of a superposition of plane waves a legitimate solution of the wave equation. In order to overcome this difficulty we introduce the so called ikonal approximation (this is usually called in the mathematicians jargon "the WKB approximation"). The representation of the phase as a line integral in a four-dimensional configuration space is discussed, and the Hamiltonian ray equations are derived as a solution of the dispersion equation in a state space representation. The equations are shown to satisfy the uniqueness conditions prescribed by the phase integral.

The Fermat Principle and its Relativistic Connotations

The Fermat principle is usually stated as saying that the ray will traverse the distance between two points in extremal (usually minimal) time. For media not varying in time. The four-dimensional Fermat principle involves media varying in space and time. The variational priciple (Syngh 1954) for the extended Fermat principle states that the variation of the phase vanishes. It is shown that the Euler-Lagrange equations of this variational principle are the Hamilton equations, and that the extremized parameter is the relativistic proper time.

Application to Ray Propagation in Lossy Media

Another application which invokes questions of Lorentz invariance and relativistic transformations, coupled with analyticity of functions, is the question of ray propagation in lossy media. At a first glance this appears as a very unlikely candidate for this role, but there are some fundamental questions involved, tied in with relativistic problems which posses important engineering implications. In lossy media the dispersion equation is complex. Consequently the group velocity will become complex too, in general, in turn implying complex space and time. The main problem is that numerous models are feasible for extending the group velocity to the present case. All the models define group velocities which reduce to the conventional definition in lossless media, but the physical consequences vary from one definition to another. There are essentially two main schools of thought: Some researchers do not experience any difficulty in continuing the concepts of a real group velocity, and real space and time, into the complex domain. This introduces conceptual difficulties of dealing with complex space and time, and having to come to terms with a complex group velocity for which no physical explanation can be found. The other group of researchers advocates using real group velocities even in the presence of lossy media. The present model belongs to this class. The difficulty with many of these models is that they do not maintain analyticity. We also mention in passing that analyticity has a lot to do with causality, e.g., *via* the Kramers-Kronig relations, and also the fact that the zeroes of the dispersion equation are the poles determining the free space Green function for the medium at hand. The following model offers a definition for the phase and group velocity which keeps the group velocity simultaneously real (i.e., confined to the real axes of the relevant complex variables complex planes) and analytic, therefore commensurate with the pertinent relativistic transformation formulas.

Formulas (mentioned during the talk)

$$\nabla\times\mathbf{E} = -\partial_t\mathbf{B} - \mathbf{j}_m, \quad \nabla\times\mathbf{H} = \partial_t\mathbf{D} + \mathbf{j}_e$$
$$\nabla\cdot\mathbf{D} = \rho_e, \quad \nabla\cdot\mathbf{B} = \rho_m \qquad (1)$$

$$\mathbf{f}_e = q_e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad (2)$$

$$\mathbf{f}_m = q_m(\mathbf{H}-\mathbf{v}\times\mathbf{D}) \quad (3)$$

$$\nabla'\times\mathbf{E}' = -\partial_{t'}\mathbf{B}' - \mathbf{j}_m', \quad \nabla'\times\mathbf{H}' = \partial_{t'}\mathbf{D}' + \mathbf{j}_e'$$
$$\nabla'\cdot\mathbf{D}' = \rho_e', \quad \nabla'\cdot\mathbf{B}' = \rho_m' \qquad (4)$$

$$\mathbf{x}' = \tilde{\mathbf{U}}\cdot(\mathbf{x} - \mathbf{v}t),\ t' = \gamma(t - \mathbf{x}\cdot\mathbf{v}/c^2) \quad (5)$$

$$\gamma = (1-\beta^2)^{-1/2}, \quad \tilde{\mathbf{U}} = \tilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}},$$

$$\hat{\mathbf{v}} = \mathbf{v}/v, \quad v = |\mathbf{v}|, \quad \beta = v/c \qquad (6)$$

$$\nabla' = \tilde{\mathbf{U}}\cdot(\nabla + \mathbf{v}\partial_t/c^2), \quad \partial_{t'} = \gamma(\partial_t + \mathbf{v}\cdot\nabla) \quad (7a)$$

$$\partial_{\mathbf{x}'} = \tilde{\mathbf{U}}\cdot(\partial_{\mathbf{x}} + \mathbf{v}\partial_t/c^2), \quad \partial_{t'} = \gamma(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{x}}) \quad (7b)$$

$$\mathbf{E}' = \tilde{\mathbf{V}}\cdot(\mathbf{E} + \mathbf{v}\times\mathbf{B}), \quad \mathbf{B}' = \tilde{\mathbf{V}}\cdot(\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2)$$
$$\mathbf{D}' = \tilde{\mathbf{V}}\cdot(\mathbf{D} + \mathbf{v}\times\mathbf{H}/c^2), \quad \mathbf{H}' = \tilde{\mathbf{V}}\cdot(\mathbf{H} - \mathbf{v}\times\mathbf{D})$$
$$\mathbf{j}_e' = \tilde{\mathbf{U}}\cdot(\mathbf{j}_e - \rho_e\mathbf{v}), \quad \rho_e' = \gamma(\rho_e - \mathbf{j}_e\cdot\mathbf{v}/c^2)$$
$$\mathbf{j}_m' = \tilde{\mathbf{U}}\cdot(\mathbf{j}_m - \rho_m\mathbf{v}), \quad \rho_m' = \gamma(\rho_m - \mathbf{j}_m\cdot\mathbf{v}/c^2)$$
$$\tilde{\mathbf{V}} = \gamma\tilde{\mathbf{I}} + (1-\gamma)\hat{\mathbf{v}}\hat{\mathbf{v}} \qquad (8)$$

$$\mathbf{X} = (\mathbf{x}, ict), \quad \partial_{\mathbf{X}} = \left(\partial_{\mathbf{x}}, -\frac{i}{c}\partial_t\right) \quad (9)$$

$$\mathbf{X}\cdot\mathbf{X} = x^2 - c^2t^2 = \mathbf{X}'\cdot\mathbf{X}' = x'^2 - c^2t'^2 = \text{constant} \quad (10)$$
$$\mathbf{J}_e = (\mathbf{j}_e, ic\rho_e), \quad \mathbf{J}_m = (\mathbf{j}_m, ic\rho_m) \quad (11)$$

$$f(\mathbf{X}) = \int f(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}}\,d^4\mathbf{K} \quad (12) \qquad \partial_{\mathbf{X}} f(\mathbf{X}) = \int i\mathbf{K}\, f(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{X}}\,d^4\mathbf{K} \quad (13)$$

$$\mathbf{k}' = \tilde{\mathbf{U}}\cdot(\mathbf{k} - \mathbf{v}\omega/c^2), \quad \omega' = \gamma(\omega - \mathbf{v}\cdot\mathbf{k}) \quad (14)$$

$$f(\mathbf{K}) = \frac{1}{(2\pi)^4}\int f(\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}}\,d^4\mathbf{X} \quad (15)$$
$$\partial_{\mathbf{K}} = (\partial_{\mathbf{k}}, -ic\partial_\omega)$$

$$\partial_{\mathbf{k}'} = \tilde{\mathbf{U}}\cdot(\partial_{\mathbf{k}} + \mathbf{v}\partial_\omega), \quad \partial_{\omega'} = \gamma(\partial_\omega + c^{-2}\mathbf{v}\cdot\partial_{\mathbf{k}}) \quad (16)$$

$$\int \mathbf{E}'(\mathbf{K}')e^{i\mathbf{K}'\cdot\mathbf{X}'}d^4\mathbf{K}' = \int \tilde{\mathbf{V}}\cdot[\mathbf{E}(\mathbf{K}) + \mathbf{v}\times\mathbf{B}(\mathbf{K})]e^{i\mathbf{K}\cdot\mathbf{X}}d^4\mathbf{K} \qquad (17)$$

$$d^4\mathbf{K}' = |\partial_{\mathbf{K}}\mathbf{K}'|\, d^4\mathbf{K} = |\tilde{\mathbf{W}}|\, d^4\mathbf{K} = d^4\mathbf{K} \quad (18)$$

$$\mathbf{E}'(\mathbf{K}') = \tilde{\mathbf{V}}\cdot[\mathbf{E}(\mathbf{K}) + \mathbf{v}\times\mathbf{B}(\mathbf{K})] \quad (19)$$

$$d^4\mathbf{X} = d^4\mathbf{X}' \quad (20)$$

$$\partial_{\mathbf{X}}\cdot\partial_{\mathbf{X}} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \quad (21)$$

$$\partial_{\mathbf{X}}\cdot\mathbf{J}_e = \nabla\cdot\mathbf{j}_e + \partial_t\rho_e = 0 \quad (22)$$

$$\partial_{\mathbf{K}}\cdot\mathbf{J}_e(\mathbf{K}) = \partial_{\mathbf{k}}\cdot\mathbf{j}_e(\mathbf{k},\omega) + c^2\partial_\omega\rho_e(\mathbf{k},\omega) = \text{constant} \quad (23)$$

$$\mathbf{K}\cdot\mathbf{J}_e(\mathbf{K}) = \mathbf{K}\cdot\mathbf{J}_m(\mathbf{K}) = 0 \quad (24)$$

$$\mathbf{X},\ \partial_{\mathbf{X}},\ \mathbf{J}_e,\ \partial_{\mathbf{J}_e},\ \mathbf{J}_m,\ \partial_{\mathbf{J}_m},\ \mathbf{K},\ \partial_{\mathbf{K}} \quad (25)$$

$$c^2B^2 - E^2 = \text{constant},\ H^2 - c^2D^2 = \text{constant},$$
$$\mathbf{B}\cdot\mathbf{E} = \text{constant},\ \mathbf{H}\cdot\mathbf{D} = \text{constant},$$
$$\mathbf{B}\cdot\mathbf{H} - \mathbf{E}\cdot\mathbf{D} = \text{constant},\ c^2\mathbf{B}\cdot\mathbf{D} + \mathbf{E}\cdot\mathbf{H} = \text{constant} \quad (26)$$

$$\begin{aligned} &i\mathbf{k}\times\mathbf{E}_e = i\omega\mathbf{B}_e &&,\ i\mathbf{k}\times\mathbf{E}_m = i\omega\mathbf{B}_m - \mathbf{j}_m \\ &i\mathbf{k}\times\mathbf{H}_e = -i\omega\mathbf{D}_e + \mathbf{j}_e &&,\ i\mathbf{k}\times\mathbf{H}_m = -i\omega\mathbf{D}_m \\ &i\mathbf{k}\cdot\mathbf{D}_e = \rho_e &&,\ i\mathbf{k}\cdot\mathbf{D}_m = 0 \\ &i\mathbf{k}\cdot\mathbf{B}_e = 0 &&,\ i\mathbf{k}\cdot\mathbf{B}_m = \rho_m \end{aligned} \quad (27)$$

$$\begin{aligned} &\mathbf{j}_e \leftrightarrow -\mathbf{j}_m,\ \rho_e \leftrightarrow -\rho_m \\ &\mathbf{E}_e \leftrightarrow \mathbf{H}_m,\ \mathbf{H}_e \leftrightarrow \mathbf{E}_m \\ &\mathbf{B}_e \leftrightarrow -\mathbf{D}_m,\ \mathbf{D}_e \leftrightarrow -\mathbf{B}_m \\ &\mathbf{A}_e \leftrightarrow -\mathbf{A}_m,\ \phi_e \leftrightarrow -\phi_m \\ &\mathbf{\Phi}_e \leftrightarrow -\mathbf{\Phi}_m \end{aligned} \quad (28)$$

$$\begin{aligned} &\mathbf{B}_e = i\mathbf{k}\times\mathbf{A}_e &&,\ \mathbf{D}_m = i\mathbf{k}\times\mathbf{A}_m \\ &\mathbf{E}_e = -i\mathbf{k}\phi_e + i\omega\mathbf{A}_e &&,\ \mathbf{H}_m = i\mathbf{k}\phi_m - i\omega\mathbf{A}_m \\ &\mathbf{\Phi}_e = \left(\mathbf{A}_e, \frac{i}{c}\phi_e\right) &&,\ \mathbf{\Phi}_m = \left(\mathbf{A}_m, \frac{i}{c}\phi_m\right) \end{aligned} \quad (29)$$

$$\mathbf{A}_e' = \tilde{\mathbf{U}}\cdot(\mathbf{A}_e - \mathbf{v}\phi_e/c^2)\ ,\ \mathbf{A}_m' = \tilde{\mathbf{U}}\cdot(\mathbf{A}_m - \mathbf{v}\phi_m/c^2)$$
$$\phi_e' = \gamma(\phi_e\omega - \mathbf{v}\cdot\mathbf{A}_e)\ ,\ \phi_m' = \gamma(\phi_m\omega - \mathbf{v}\cdot\mathbf{A}_m) \quad (30)$$

$$\mathbf{K}\cdot\mathbf{\Phi}_e = \mathbf{k}\cdot\mathbf{A}_e - \frac{\omega}{c^2}\phi_e \quad (31)$$

$$\frac{\partial}{\partial X_i}\frac{\partial a}{\partial X_j} - \frac{\partial}{\partial X_j}\frac{\partial a}{\partial X_i} = 0\ ,\ i,j = 1, 2, 3, 4 \quad (32)$$

$$dS = \sqrt{d\mathbf{X}\cdot d\mathbf{X}} = \sqrt{(dx)^2 - c^2(dt)^2} \quad (33)$$

$$d\tau = dS/ic = dt\sqrt{1 - \frac{(d\mathbf{x})^2}{c^2(dt)^2}} = dt\sqrt{1 - \frac{v^2}{c^2}} = dt/\gamma \quad (34)$$

$$\mathbf{V} = \frac{d\mathbf{X}}{d\tau} = \gamma(\mathbf{v}, ic) \quad (35) \qquad \mathbf{v}' = \frac{\tilde{\mathbf{U}}\cdot(\mathbf{v}-\mathbf{v}_0)}{\gamma(1-\mathbf{v}\cdot\mathbf{v}_0/c^2)}\ ,\ \beta = v_0/c \quad (36)$$

$$\mathbf{W} = \frac{d\mathbf{V}}{d\tau} \quad (37) \qquad \mathbf{V}\cdot\mathbf{W} = \mathbf{V}\cdot\frac{d\mathbf{V}}{d\tau} = \frac{1}{2}\frac{d}{d\tau}|\mathbf{V}|^2 = -\frac{1}{2}\frac{d}{d\tau}c^2 = 0 \quad (38)$$

$$\mathbf{F} = m\mathbf{W} \ (39) \quad \mathbf{V}\cdot\mathbf{F} = 0 \ (40) \quad \mathbf{F} = (\gamma\mathbf{f}, i\gamma q_e\mathbf{v}\cdot\mathbf{E}/c) \ (41)$$

$$\mathbf{F}\cdot\mathbf{F} = \mathbf{F}'\cdot\mathbf{F}' = \mathbf{f}'\cdot\mathbf{f}' = q_e^2\mathbf{E}'\cdot\mathbf{E}' \quad (42)$$

$$\mathbf{D}'(\mathbf{K}') = \tilde{\boldsymbol{\varepsilon}}'(\mathbf{K}')\cdot\mathbf{E}'(\mathbf{K}')$$
$$\mathbf{B}'(\mathbf{K}') = \tilde{\boldsymbol{\mu}}'(\mathbf{K}')\cdot\mathbf{H}'(\mathbf{K}') \quad (43)$$

$$\mathbf{D}'(t') = \int_{-\infty}^{t'} \varepsilon'(\tau')\cdot\mathbf{E}'(t' - \tau')\, d\tau' \quad (44)$$

$$\mathbf{D}'(\mathbf{X}') = \int_{\mathbf{\Xi}_1}^{\mathbf{\Xi}_2} \tilde{\boldsymbol{\varepsilon}}'(\mathbf{\Xi}')\cdot\mathbf{E}'(\mathbf{X}' - \mathbf{\Xi}')\, d^4\mathbf{\Xi}' \quad (45)$$

$$\mathbf{D} + \mathbf{v}\times\mathbf{H}/c^2 = \tilde{\boldsymbol{\varepsilon}}_v\cdot(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$
$$\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2 = \tilde{\boldsymbol{\mu}}_v\cdot(\mathbf{H} - \mathbf{v}\times\mathbf{D})$$
$$\tilde{\boldsymbol{\varepsilon}}_v = \tilde{\mathbf{V}}^{-1}\cdot\tilde{\boldsymbol{\varepsilon}}'\cdot\tilde{\mathbf{V}}\ ,\ \tilde{\boldsymbol{\mu}}_v = \tilde{\mathbf{V}}^{-1}\cdot\tilde{\boldsymbol{\mu}}'\cdot\tilde{\mathbf{V}} \quad (46)$$

$$i\mathbf{k}'\times\mathbf{E}' = i\omega'\mathbf{B}' - \mathbf{j}_m',\ i\mathbf{k}'\cdot\mathbf{B}' = 0$$
$$i\mathbf{k}'\times\mathbf{H}' = -i\omega'\mathbf{D}' + \mathbf{j}_e',\ i\mathbf{k}'\cdot\mathbf{D}' = 0 \quad (47a)$$

$$\mathbf{j}_e' = \tilde{\boldsymbol{\sigma}}_e'\cdot\mathbf{E}',\ \mathbf{j}_m' = \tilde{\boldsymbol{\sigma}}_m'\cdot\mathbf{H}' \quad (47)$$

$$\mathbf{k}'\times\mathbf{E}' - \omega'\tilde{\boldsymbol{\mu}}'^{\dagger}\cdot\mathbf{H}' = 0,\ \mathbf{k}'\times\mathbf{H}' + \omega'\tilde{\boldsymbol{\varepsilon}}'^{\dagger}\cdot\mathbf{E}' = 0$$
$$\mathbf{k}'\cdot\mathbf{D}' = 0,\ \mathbf{k}'\cdot\mathbf{B}' = 0 \quad (47b)$$

$$F'(\mathbf{K}') = 0 \ (48a) \quad F'(\mathbf{K}'[\mathbf{K}]) = 0 = F(\mathbf{K}) \ (48b)$$

$$f(\mathbf{X}) = \int \delta(F) f(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}}\, d^4\mathbf{K} = \int f(\mathbf{k},\omega[\mathbf{k}]) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k})t}\, d^3\mathbf{k} \quad (49)$$

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + \partial_{\mathbf{k}}\omega(\mathbf{k})\big|_{\mathbf{k}=\mathbf{k}_0}\cdot(\mathbf{k}-\mathbf{k}_0) = \omega_0 + \mathbf{v}_g\cdot(\mathbf{k}-\mathbf{k}_0) \quad (50)$$

$$f(\mathbf{x},t,\mathbf{k}_0) = e^{i\mathbf{k}_0\cdot\mathbf{x} - i\omega_0 t}\int f(\mathbf{k},\omega[\mathbf{k}]) e^{i(\mathbf{k}-\mathbf{k}_0)\cdot[\mathbf{x} - \mathbf{v}_g(\mathbf{k}_0)t]} d^3\mathbf{k} \quad (51)$$

$$A(\mathbf{X})e^{i\theta(\mathbf{x})},\ \partial_{\mathbf{X}}\theta(\mathbf{X}) = \mathbf{K}(\mathbf{X}) \quad (52)$$

$$\phi(\mathbf{x}) = \int_{\phi(\mathbf{x}_0)}^{\phi(\mathbf{x})} d\phi\ ,\ \phi(\mathbf{x}_0) = 0 \quad (53) \qquad \phi(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\boldsymbol{\xi})\cdot d\boldsymbol{\xi} \quad (54)$$

$$\theta(\mathbf{X}) = \int_{\theta(\mathbf{X}_0)}^{\theta(\mathbf{X})} d\theta = \int_{\mathbf{X}_0}^{\mathbf{X}} \partial_{\mathbf{X}}\theta(\mathbf{X})\cdot d\mathbf{X} = \int_{\mathbf{X}_0}^{\mathbf{X}} \mathbf{K}(\mathbf{X})\cdot d\mathbf{X} \quad (55)$$

$$\partial_{\mathbf{x}}\times\mathbf{k}(\mathbf{x}, t) = 0,\ \partial_t\mathbf{k}(\mathbf{x}, t) + \partial_{\mathbf{x}}\omega(\mathbf{x}, t) = 0 \quad (56)$$

$$\mathbf{D}(\mathbf{K}, \mathbf{X}) = \tilde{\varepsilon}(\mathbf{K}, \mathbf{X})\cdot\mathbf{E}(\mathbf{K}, \mathbf{X})$$
$$\mathbf{B}(\mathbf{K}, \mathbf{X}) = \tilde{\mu}(\mathbf{K}, \mathbf{X})\cdot\mathbf{H}(\mathbf{K}, \mathbf{X}) \quad (57) \qquad F(\mathbf{K}, \mathbf{X}) = 0 \quad (58)$$

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial\mathbf{K}}\cdot\frac{d\mathbf{K}}{d\tau} + \frac{\partial F}{\partial\mathbf{X}}\cdot\frac{d\mathbf{X}}{d\tau} = 0 \quad (59)$$

$$\frac{d\mathbf{X}}{d\tau} = \lambda(\tau)\frac{\partial F}{\partial\mathbf{K}}$$
$$\frac{d\mathbf{K}}{d\tau} = -\lambda(\tau)\frac{\partial F}{\partial\mathbf{X}} \quad (60) \qquad \lambda(\tau) = ic\Big/\sqrt{\frac{\partial F}{\partial\mathbf{K}}\cdot\frac{\partial F}{\partial\mathbf{K}}} \quad (61)$$

$$\begin{aligned} \mathbf{v}_g &= \frac{d\mathbf{x}}{dt} = -\frac{\partial F/\partial\mathbf{k}}{\partial F/\partial\omega} \\ \frac{d\mathbf{k}}{dt} &= \frac{\partial F/\partial\mathbf{x}}{\partial F/\partial\omega} \\ \frac{d\omega}{dt} &= -\frac{\partial F/\partial t}{\partial F/\partial\omega} \end{aligned} \quad (62)$$

$$0 = \delta\theta(\mathbf{X}) = \delta\int_{\mathbf{X}_0}^{\mathbf{X}_1} d\theta(\mathbf{X}) = \delta\int_{\mathbf{X}_0}^{\mathbf{X}_1} \mathbf{K}(\mathbf{X})\cdot d\mathbf{X} \quad (63a)$$

$$0 = \delta\theta(\mathbf{X}) = \delta\int_{\mathbf{X}_0}^{\mathbf{X}_1}\left\{\mathbf{K}(\mathbf{X}[\tau])\cdot\frac{d\mathbf{X}[\tau]}{d\tau} - \lambda(\tau)F(\mathbf{K}[\tau], \mathbf{X}[\tau])\right\} d\tau \quad (63b)$$

$$0 = \int_{\mathbf{X}_0}^{\mathbf{X}_1}\left\{\delta\mathbf{K}\cdot\frac{d\mathbf{X}}{d\tau} - \delta\mathbf{X}\cdot\frac{d\mathbf{K}}{d\tau} - \lambda(\tau)\left[\frac{\partial F}{\partial\mathbf{K}}\cdot\delta\mathbf{K} + \frac{\partial F}{\partial\mathbf{X}}\cdot\delta\mathbf{X}\right]\right\} d\tau \quad (63c)$$