

SOME ASPECTS OF WAVE PROPAGATION IN QUADRATIC WEAKLY NONLINEAR MEDIA

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ABSTRACT

A theory is developed for the propagation of electromagnetic waves in quadratic weakly nonlinear media. Waves with a narrow-band spectrum are studied as well as periodic waves based upon a Volterra series formalism. Conditions of steady propagation are scrutinized. It is shown that the proper conditions are rather complicated in the present case in comparison with the single harmonic case in a cubic medium.

1. INTRODUCTION

The present discussion is confined to the case of a quadratic nonmagnetic medium which supports the processes of harmonic generation. The literature on nonlinear wave propagation associated with harmonic generation is mostly limited to discrete spectra waves [1]. However, the problem of locally continuous spectra (i.e., spectra containing local strong maxima in the vicinity of the harmonic frequencies ω_0 , $2\omega_0$, $3\omega_0$, etc.) may be of special interest since it allows us to tackle wave packet propagation in nonlinear media. One interesting aspect is stable pulse propagation (solitons). This effect is known in cubic media [2]; however, a multi-harmonic regime essentially complicates the situation. The problem of steady state solution also arises for discrete-spectrum waves. It is known [1] that a harmonic solution exists in a cubic medium. Steady state multi-harmonic solutions were discussed in [3], [4]. In the present paper we analyze the existence of such type solutions in two aspects: the existence of such a state as a phenomenon and the possibility of attaining it from an initial state as a result of an evolution. Spectrum of new-generated harmonics has been discussed in [5], [6], [7]. Our analysis is based on the general description by Volterra-series formalism [8], [9]. The approach developed here facilitates analyzing the dynamic behavior of harmonic spectra and wave-packet envelopes, especially their asymptotical behavior at a long distance from initial position.

The present discussion concerns only media possessing temporal dispersion.

2. GENERAL THEORY

The approach developed here allows treating a class of nonlinear wave propagation problems where the nonlinear effects are weak. The nonlinearity is restricted to dielectric properties and hence all nonlinear effects are determined by the terms in the Maxwell equations containing an electric induction vector \mathbf{D}

$$\nabla \times \nabla \times \mathbf{E} + \mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2} = 0 \quad (1)$$

If nonlinearity is weak this vector may be represented in terms of a Volterra series [3], [4]

$$\mathbf{D} = \mathbf{D}^{(1)} + \mathbf{D}^{(2)} + \mathbf{D}^{(3)} + \dots \quad (2)$$

where every term in the right hand side is a functional of the nonlinear order defined by its upper index

$$\begin{aligned} D_n^{(p)}(\rho, t) = & \int_{v_1} \dots \int_{v_p} \int_0^\infty \dots \int_0^\infty \epsilon_{n_1 \dots n_p}^{(p)}(\tau_1, \dots, \tau_p, \mathbf{r}_1, \dots, \mathbf{r}_p) \\ & \times E_{n_1}(\mathbf{r}_1 - \rho, t - \tau_1) \dots E_{n_p}(\mathbf{r}_p - \rho, t - \tau_p) d\tau_1 \dots d\tau_p dV_1(\mathbf{r}_1) \dots dV_p(\mathbf{r}_p) \end{aligned} \quad (3)$$

For a quadratic medium we may ignore the contribution of all the terms in (1) with order higher than two. Consider a process starting (at an initial surface, say $y=0$) with a plane wave

$$\mathbf{E} = \mathbf{e}_1 E_1 e^{i(\omega t - ky)} \quad (4)$$

where \mathbf{e}_1 is a unit polarization vector and the complex amplitude E_1 may be a time dependent function which is slow-varying compared to the period $2\pi/\omega$. The nonlinear term in (1) prescribes the appearance of new harmonic frequencies and therefore a solution for the electric field is sought in the form [3], [4], [8], [9]

$$\mathbf{E} = \sum_{n=-\infty}^{\infty} \mathbf{e}_n E_n e^{in(\omega t - ky)} \quad (5)$$

where the complex amplitudes E_n are now slow-varying functions in space and possibly time-domain.

3. DISCRETE-SPECTRUM WAVES

Consider first a discrete-spectrum case. It is known [1] that in a cubic medium equation (1) may be satisfied by the solution (4) with

$$k^2 = k_0^2 + \eta |E_1|^2 \quad (6)$$

where k_0 is the solution of the linear dispersion equation and the nonlinear coefficient η is determined by the third order nonlinear dielectric permittivity. Try to obtain such a type solution for a quadratic medium. Then the amplitudes E_n in (5) are constant and the proper equation for them is

$$k^2 E_m - \mu_0 \omega^2 \left(\varepsilon_{Mm}^{(1)} E_m + \sum_{n_1+n_2=m} \varepsilon_{Mmn_1n_2}^{(2)} E_{n_1} E_{n_2} \right) = 0 \quad (7)$$

where the new scalar parameters are introduced

$$\varepsilon_{Mmn_1n_2}^{(2)} = \frac{1}{\cos^2 \alpha_m} \varepsilon_{j_1 j_1 j_2}^{(p)}(m\omega, n_1\omega, n_2\omega) e_{mj_1} e_{n_1 j_1} e_{n_2 j_2} \quad (8)$$

M denotes "modified", $p=1,2$, α_m is the angle between k and e_m , and the ε -function on the right hand side is the Fourier-transform of the corresponding ε -function in (3). Analyzing (7), one can see that, by introducing the effective parameters [4], [5] (denoted by a bar)

$$\bar{\varepsilon}_{Mn}^{(2)}(n\omega, E_1, E_2, \dots) = \frac{1}{E_n^2} \sum_{n_1+n_2=n} \varepsilon_{Mmn_1n_2}^{(2)}(n_1\omega, n_2\omega) E_{n_1} E_{n_2} \quad (9)$$

these equations may be reduced to the form

$$k^2 E_m - \mu_0 \omega^2 (\varepsilon_{Mm}^{(1)} - \bar{\varepsilon}_{Mm}^{(2)} E_m) E_m = 0 \quad (10)$$

which looks simpler since it contains only a single harmonic. The bar parameters $\bar{\varepsilon}$ in (9) depend on ratios of harmonics and therefore may be treated as constants if $E_m/E_n \approx \text{constant}$. However, the parameters (9) are indeed unknown. How can we then work with equations (10) under such circumstances? This question is addressed below.

We consider separately two different hypothetical situations:

- (a) $|E_1| \gg |E_2| \gg \dots \gg |E_n| \gg \dots$;
- (b) $|E_{n+1}|/|E_n| = 1 - \Delta_n$, where $0 < \Delta_n \ll 1$

3.1. Case (a)

Consider equations (8). The hierarchy assumed allows truncating them, retaining a finite number of harmonics only. Therefore an inductive approach may be used, i.e.:

- 1) N harmonics (starting from $N=2$) are retained;
- 2) in the next step $N+1$ harmonics are retained and the previous solution for the N first ones is treated as unperturbed.

Introduce the set of parameters $\{\gamma_{n;k}\}$

$$\gamma_{n;k} = \frac{\varepsilon_{M1}^{(1)} - \varepsilon_{Mn}^{(1)}}{4\varepsilon_{Mn;k,n-k}^{(2)} \rho_1} \quad (11)$$

where $\rho_1 = |E_1|$. These parameters reveal the interplay of three factors: dispersion nonlinearity and wave power. It can be shown that the hierarchy is provided by the condition $\gamma_{n;k} \gg 1$, which implies that dispersion is an essential factor and wave power is rather small. Then the steady state solution may be expressed as

$$\frac{\rho_n}{\rho_1} = \sum_{k=1}^{n-1} a_{n-k;k} \frac{1}{\gamma_{n;k}} \prod_{m=1}^{n-k-1} \frac{1}{\gamma_{n-k;m}} \prod_{j=1}^{k-1} \frac{1}{\gamma_{k;j}} \quad (12)$$

where $a_{n;k}$ are certain coefficients defined from (7). The next step is to examine conditions which provide the

steady state solution (12) to be attained in an evolution. As far as amplitudes in (5) may be considered as slowly varying functions, the proper dynamic equation for amplitudes take the form

$$ik \frac{dE_m}{dy} + \left(\frac{m}{2} k^2 - \mu_0 \omega^2 \varepsilon_{Mm}^{(1)} \right) E_m - \mu_0 \omega^2 \frac{m}{2} \sum_{n_1+n_2=m} \varepsilon_{Mmn_1n_2}^{(2)} E_{n_1} E_{n_2} = 0 \quad (13)$$

It can be shown that even in the simplest case $N=2$ the proper condition fixes the necessary amplitude

$$\rho_1 = [(\kappa^2/\eta^2 - 2)/8]^{1/2} \quad (14)$$

where k and h are coefficients determined by the dispersion and nonlinear properties respectively. Thus, if such a state can be attained it proves to be unstable.

3.2. Case (b)

Introduction of the bar parameters (9) is justified if $\rho_n/\rho_m \approx \text{const}$. As may be shown, this requires γ -parameters to be very small. Since we deal with rather weak fields only, the smallness of γ means a very weak dispersion. This circumstance motivates the investigation of media without, or with very weak, dispersion.

Note that in such cases, a truncation procedure is inadequate. Nevertheless, if we should truncate the equations, we obtain the following numerical results

- $N=2$, $\rho_2/\rho_1 = 0.707$;
- $N=3$, $\rho_3/\rho_2 = 0.743$;
- $N=4$, $\rho_4/\rho_3 = 0.888$;
- $N=5$, $\rho_5/\rho_4 = 0.919$;
- $N=6$, $\rho_6/\rho_5 = 0.940$.

They show that for rather large N , case (b) is indeed realized. For the nondispersive case, (7) becomes

$$\bar{E}_n = \sum_{m+k=n} \bar{E}_m \bar{E}_k \quad (15)$$

$$\text{where } \bar{E}_n = \frac{\mu_0 \varepsilon_M^{(2)}}{\frac{1}{\sqrt{2}} - \mu_0 \varepsilon_M^{(1)}} E_n \quad (16)$$

$$\text{and } \varepsilon_M^{(2)} = \varepsilon_{M2,1,1}^{(2)}$$

Consider (15) for very large N . It may be shown that for large $n \leq N$ the solution of (15) satisfies the following estimation (the proof is omitted here)

$$\frac{E_{n+1}}{E_n} \sim 1 - \frac{1}{n^{1-\beta}} \quad (17)$$

where $0 < \beta < 1$.

Using this estimation we can implement an efficient algorithm for solving steady state equations in the form (10) for weakly dispersive media: relation (17) provides the

initial evaluation for effective dielectric parameters (9) and renders them thus applicable for calculations. This algorithm is described by the following steps.

1st step: Parameters (9) are evaluated by means of (17).

2nd step: Considering the fundamental harmonic amplitude as a known we find from (10) the phase velocity which has in a steady state, the same value for all harmonics

$$v^2 = \frac{\omega^2}{k^2} = \frac{1}{\mu_0(\epsilon_{M1}^{(1)} - \bar{\epsilon}_{M1}^{(2)} E_1)} \quad (18)$$

3rd step: Directly from equations (10) we find the n-th harmonic amplitude for any n

$$E_n = \frac{\frac{1}{v^2} - \mu_0 \epsilon_{Mn}^{(1)}}{\bar{\epsilon}_{Mn}^{(2)}} = \mu_0 \frac{\epsilon_{M1}^{(1)} - \epsilon_{Mn}^{(1)} - \bar{\epsilon}_{M1}^{(2)} E_1}{\bar{\epsilon}_{Mn}^{(2)}} \quad (19)$$

4th step: Determine more accurate evaluation for parameters (9) by means of the amplitudes found.

5th step: Iterate to find more accurate values.

This solution proves to be stable; however, its achievability in an evolution process still remains an open question.

4. CONTINUOUS NARROW-BAND SPECTRUM

Consider a case, where the input field changes slowly in the time-domain. Then the amplitudes $E_n(y, t)$ in (5) are slowly varying functions in time and space domain. The amplitudes E_n are represented by their Fourier spectrum

$$E_n(y, t) = \int_{-\infty}^{\infty} E_n(\Omega_n, y) e^{i[\Omega_n t - \Delta K_n(\Omega_n) y]} d\Omega_n \quad (20)$$

Here $\Omega_n = \omega - n\omega_0$ and $\Delta K_n(\Omega_n) = k(\omega) - k_n$, the constant factor before the integral is omitted and a Gaussian type spectrum is presumed

$$E_n(\Omega_n, y) = E_{n0}(y) e^{-a_n(y)\Omega_n^2 - b_n(y)\Omega_n} \quad (21)$$

The parametrization of the spectrum by the set of y-dependent parameters $\{a_n, b_n\}$ was introduced in [9].

Henceforth, we assume $a_n \gg (1/n\omega_0)^2$, i.e., the n-th harmonic spectrum is sufficiently narrow, b_n is a measure of the spectrum asymmetry and $a_n \gg |b_n|$. The following condition at some initial plane $y=0$ are also imposed

$$b_n(0) = 0, \quad a_n(0) = a/nl \quad (22)$$

which indicates that interaction of locally continuous spectra leads to broadening. We intend, however, to deal with waves whose spectrum is narrow with respect to ω_0 . Although (5) spans an infinite range n, physical media act as a low pass filter, effectively limiting n to some upper N. This is due to the fact that at high frequencies $\epsilon/\epsilon_0 \rightarrow 1$, hence nonlinear effects diminish. Our criterion is taken as $a_N \gg (1/\omega_0)^2$.

As previously, assume that (2) converges and therefore can be approximated by the significant leading terms only. For quadratic media $p=2$ is retain and all higher nonlinear orders

are ignored. Linear permittivity for each harmonic is expanded in a Taylor series inside the band-width of the proper spectrum line and the first order dispersion correction is retained [7]

$$\epsilon_{jm}^{(1)}(n\omega_0 + \Omega_n) = \epsilon_{jm}^{(1)}(n\omega_0) + \left. \frac{d\epsilon_{jm}^{(1)}(\omega)}{d\omega} \right|_{n\omega_0} \Omega_n$$

$$\Delta K_n(\Omega_n) = \left. \frac{dk}{d\omega} \right|_{n\omega_0} = \frac{dk_n}{d\omega} \Omega_n = s_n \Omega_n \quad (23)$$

where $1/s_n$ is recognized as the group velocity. For the nonlinear permittivity all dispersion corrections inside the band-width of a spectrum line are ignored. Substituting (5) and (23), the integration in (4) yields to the proper Fourier components

$$D_{;j}^{(1)}(\omega) = e_{n;m} \left\{ \epsilon_{jm}^{(1)}(n\omega_0) + \left. \frac{d\epsilon_{jm}^{(1)}(\omega)}{d\omega} \right|_{n\omega_0} \Omega_n \right\}$$

$$\times E_{n0}(y) e^{-a_n(y)\Omega_n^2 - b_n(y)\Omega_n} e^{-i\Omega_n s_n y} e^{-ik_n y} \quad (24)$$

$$D_{;i}^{(2)}(\omega) = \sum_{p+q=n} \epsilon_{ijm}^{(2)}(p\omega_0; q\omega_0) e_{p;j} e_{q;m}$$

$$\times R_{pq} e^{-i(k_p + k_q)y} \quad (25)$$

where n is the integer part of ω/ω_0 and R_{pq} is

$$R_{pq} = \sqrt{\frac{\pi}{a_p + a_q}} E_{p0} E_{q0} \exp \left\{ -\frac{a_p b_p + a_q b_q}{a_p + a_q} \Omega_n \right.$$

$$- \frac{a_p a_q}{a_p + a_q} \Omega_n^2 + i \frac{b_p - b_q}{2(a_p + a_q)} (s_p - s_q) y$$

$$\left. - i \frac{a_p s_q + a_q s_p}{a_p + a_q} \Omega_n y - \frac{(s_p - s_q)^2}{4(a_p + a_q)} y^2 \right\} \quad (26)$$

The value $a_p a_q / (a_p + a_q)$ corresponds to a_n and this expression justifies condition (22) for a_n .

Insert the approximation $a_p(y) = a/|p|$ into (26). Substitute (24)-(26) into (1) and set $\omega = n\omega_0$ in order to obtain equations for amplitudes

$$i2k \frac{\partial}{\partial y} E_{n0} + n[k^2 - \mu_0 \omega_0^2 \epsilon_{Mn}^{(1)}] E_{n0}$$

$$- \mu_0 n \omega_0^2 \sqrt{\frac{\pi}{a}} \sum_{p+q=n} \sqrt{\frac{|pq|}{|p| + |q|}}$$

$$\times \epsilon_{Mn;p;q}^{(2)} E_{p0} E_{q0} e^{i\Delta k_{npq} y - \chi_{npq} y^2} = 0 \quad (27)$$

where $k_p=pk$ is used and k is the steady state value corresponding to discrete harmonics [8]

$$\Delta k_{npq} = \frac{b_p - b_q}{2(a_p + a_q)}(s_p - s_q) \quad (28)$$

$$\chi_{npq} = \frac{(s_p - s_q)^2}{4(a_p + a_q)} \quad (29)$$

Note, that in (29) $p=q$ prescribes $\chi_{npq}=0$. In (27) $p+q=n$, hence $p \neq q$ for all odd n and thus $\chi_{npq} > 0$. Consequently for odd n and large $y > y_0 = (\chi_{npq})^{-1/2}$ the nonlinear terms in (27) vanish. It follows that, in the framework of approximation $a_p(y) = a/|p|$ accepted for the nonlinear terms in (27), the difference of group velocity $1/s_p$ for different harmonics results in nonlinear interaction disappearance at a large distance. This describes the tendency of spatial separation different harmonic packets possessing different group velocities, defined as $v_g = \partial\omega/\partial k$. Note also that according to (28) the phase-matching condition [8] cannot be satisfied accurately for waves with an asymmetric spectrum. Derivation with respect to Ω_n , and using the relevant approximation yields after setting $\omega = n\omega_0$ the proper equation for asymmetry spectrum parameter b_n

$$\begin{aligned} i2k \frac{db_n}{dy} + n[k^2 - \mu_0\omega_0^2 \epsilon_{Mn}^{(1)}]b_n &= in\omega_0^2 \mu_0 y \\ &\times \sqrt{\frac{\pi}{a}} \sum_{p+q=n} \sqrt{|pq|} \frac{|p|s_q + |q|s_p}{(|p| + |q|)^{3/2}} \\ &\times \epsilon_{Mn;p;q}^{(2)} \frac{E_{p0}(y)E_{q0}(y)}{E_{n0}(y)} e^{i\Delta k_{npq}y - \chi_{npq}y^2} \end{aligned} \quad (30)$$

Repeated derivation, and substituting from (27) we obtain the equation for the spectrum band-width a_n

$$\begin{aligned} i2k \frac{da_n}{dy} &= \mu_0 \sqrt{\frac{\pi}{a}} \sum_{p+q=n} \sqrt{\frac{|pq|}{|p| + |q|}} \\ &\times \epsilon_{Mn;p;q}^{(2)} \frac{E_{p0}(y)E_{q0}(y)}{E_{n0}(y)} e^{i\Delta k_{npq}y - \chi_{npq}y^2} \times \\ &\left\{ a\omega_0^2 \left(\frac{n}{|p| + |q|} - 1 \right) + i\omega_0 \frac{|p|s_q + |q|s_p}{|p| + |q|} y \right. \\ &\left. + \frac{n\omega_0^2}{2} \left(\frac{|p|s_q + |q|s_p}{|p| + |q|} \right)^2 y^2 - \frac{1}{n} - 1 \right\} \end{aligned} \quad (31)$$

Note that subject to the present approximations the equations for b_n and a_n proves to be linear. This is the essential feature of the present approach.

4.1. Case with Two Harmonics

To investigate these equations in more detail consider an example where only two harmonics are assumed to be involved in the process. We now have $\chi_{211}=0$, $\Delta k_{211}=0$, $\chi_{12-1}=\chi$, $\Delta k_{12-1}=\Delta k$. The proper equations are

$$\begin{aligned} i2k \frac{\partial}{\partial y} E_{10} + (k^2 - \mu_0\omega_0^2 \epsilon_{M1}^{(1)})E_{10} - 2\mu_0\omega_0^2 \\ \times \sqrt{\frac{2\pi}{3a}} \epsilon_M^{(2)} E_{20} E_{10}^* e^{i\Delta ky - \chi y^2} = 0 \\ ik \frac{\partial}{\partial y} E_{20} + (k^2 - \mu_0\omega_0^2 \epsilon_{M2}^{(1)})E_{20} \\ - \mu_0\omega_0^2 \sqrt{\frac{\pi}{2a}} \epsilon_M^{(2)} E_{10} E_{10} = 0 \end{aligned} \quad (32)$$

where $\epsilon_M^{(2)} = \epsilon_{M2;1;1}^{(2)} = \epsilon_{M1;2;-1}^{(2)}$, and

$$\begin{aligned} \frac{da_1}{dy} &= i \frac{\mu_0}{k} \sqrt{\frac{2\pi}{3a}} \epsilon_M^{(2)} \frac{E_{20} E_{10}^*}{E_{10}} e^{i\Delta ky - \chi y^2} \times \\ &\left\{ \frac{1}{3} a\omega_0^2 - i\omega_0 \frac{2s_2 + s_1}{6} y - \frac{1}{4} \omega_0^2 \left(\frac{2s_2 + s_1}{3} \right)^2 y^2 \right\} \\ \frac{da_2}{dy} &= i \frac{\mu_0}{k} \sqrt{\frac{\pi}{2a}} \epsilon_M^{(2)} \frac{E_{10} E_{10}}{E_{20}} \\ &\times \left(\frac{1}{2} - i2\omega_0 s_1 y + 4\omega_0^2 s_1^2 y^2 \right) \end{aligned} \quad (33)$$

In this brief discussion, we concentrate on E_n and a_n . The parameter b_n will be discussed below without writing out the equation for it.

Consider the asymptotic behavior at large distance $\chi y^2 \rightarrow \infty$. It follows from (32), (33) that

$$E_1 \sim E_{1\infty} e^{iK_1 y}, \quad a_1 \sim a_{1\infty} \quad (34)$$

where $E_{1\infty}$ and $a_{1\infty}$ are constants and

$$K^2 = k^2 - \mu_0 \epsilon_{M1}^{(1)} \omega_0^2 \quad (35)$$

Thus we have

$$E_1 e^{-iky} = E_{1\infty} e^{-ik_{10}y}$$

where k_{10} is the fundamental harmonic propagation constant in the linear approximation

$$k_{10}^2 = \mu_0 \epsilon_{M1}^{(1)} \omega_0^2 \quad (36)$$

Thus in the framework of the present approximation the propagation constant of an odd harmonic takes its linear value at a large distance. Taking this into account we arbitrarily set the linear value for the wave number of an even harmonic. Then for the second harmonic we have

$$E_2 \sim E_{2\infty} e^{-ik_{20}y} - \Gamma e^{-i2k_{10}y} \quad (37)$$

where

$$\Gamma = \frac{\mu_0 \omega_0^2}{k_2(2k_{10} - k_{20})} \sqrt{\frac{\pi}{2a}} \epsilon_M^{(2)} E_{1\infty} E_{1\infty} \quad (38)$$

and

$$a_2 = a_{2\infty} - \frac{1}{2\omega_0^2} \ln \left(\frac{E_{2\infty}}{\Gamma} - e^{i\xi y} \right) + \frac{2\xi s_1}{\omega_0} \Gamma \int \frac{e^{i\xi y} y}{E_{2\infty} - \Gamma e^{i\xi y}} dy + i4\xi s_1^2 \Gamma \int \frac{e^{i\xi y} y^2}{E_{2\infty} - \Gamma e^{i\xi y}} dy \quad (39)$$

where $\xi = 2k_{10} - k_{20}$.

The analysis of asymptotical behavior here is independent of a certain initial condition. Hence we can divert our attention from initial factors and consider for an illustration a case where $\Gamma \ll E_{2\infty}$. Then (39) can be calculated in the closed form

$$a_2 \approx a_{2\infty} - \frac{1}{2\omega_0^2} \ln \frac{E_{2\infty}}{\Gamma} + \frac{2s_1 \Gamma}{E_{2\infty}} \left\{ \frac{1}{\omega_0 \xi} \left(1 - \frac{2s_1 \omega_0}{\xi} \right) - iy \frac{1}{\omega_0} \left(1 + i \frac{2s_1 \omega_0}{\xi} \right) + y^2 2s_1 \right\} e^{i\xi y} \quad (40)$$

The real part of a_2 defines the asymptotic behavior of the second harmonic wave envelope.

However, expression (40) detects that the approximation $a_p(y) = a/|p|$ in the nonlinear terms is not valid in a distant region. Indeed, it shows

$$a_2 \sim a_{2\infty} + \text{const}_1 y^2 e^{i\xi y} \quad (41)$$

and, as can be obtained in a similar way

$$b_2 \approx b_{2\infty} - \frac{1}{\omega_0} \ln \frac{E_{2\infty}}{\Gamma} + \frac{2s_1 \Gamma}{E_{2\infty}} \left(\frac{1}{\xi} - iy \right) e^{i\xi y} \quad (42)$$

$$b_2 \sim b_{2\infty} + \text{const}_2 y e^{i\xi y} \quad (43)$$

Relations (41) and (43) describe the process of the wave packet corruption. This tendency counteracts the difference in the packets group velocities and prevents the spatial separation of the packets. Indeed, insertion of (41), (43) into (28) and (29) yields

$$\begin{aligned} \text{Re}\{i\Delta ky\} &\sim \text{const}_3 \\ \text{Re}\{xy^2\} &\sim \text{const}_4 / (\cos \xi y + \text{const}_5 / y^2) \end{aligned} \quad (44)$$

and the nonlinear terms do not vanish asymptotically. In expressions (41), (43) and (44) symbols $\text{const}_1, \dots, \text{const}_5$ denote certain constant coefficients defined from the proper equations.

In any case, the polynomial term y^2 proves to be dominant for rather large y and determines the asymptotical behavior of the packets in this region.

The analysis of equations (30), (31) shows that in a distant region compression and stretch processes give way to each other depend on the phase shift between E_1 and E_2 .

In a near region (small y) the linear and nonlinear terms in (27), (30), (31) can cooperate or counteract depending on their signs. In such a region the problem of steady packet propagation can be considered if the steady state for amplitudes is attained in this region in the sense of Section 3. Note that no space limitation arises in a cubic medium for a single harmonic regime. Repeating for $D^{(3)}$ the calculations resulting in the expression for $D^{(2)}$, it may be shown a parameter analogous to χ proves to be zero in a cubic medium.

4.2. A Possible Generalization

The approach developed here may be generalized. Representation (3) means that harmonic spectra retain their Gaussian shape, whereas, strictly speaking such an assumption is not justified. For instance, the nonlinear interaction may be much more intensive in the vicinity of the fundamental harmonic central frequency. This suggests that the fundamental harmonic spectrum will be depleted in this region, leading to a double humped shape. This conjecture will have to be examined in the future. Thus it would be plausible to use a higher order polynomial

$$E_n(\Omega_n, y) = E_{no}(y) e^{-c_n(y)\Omega_n^4 - a_n(y)\Omega_n^2 - b_n(y)\Omega_n} \quad (45)$$

with the condition $c_n(y) > 0$.

REFERENCES

- [1]. L. D. Landau, E. M. Lifshitz and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.
- [2]. A. C. Newell and J. V. Moloney, *Nonlinear Optics*, John Wiley & Sons, 1992.
- [3]. D. Censor, "Ray tracing in weakly nonlinear moving media", *J. of Plasma Physics*, 16, 415-426, 1976.
- [4]. D. Censor, "Waveguide and Cavity Oscillations in the Presence of Nonlinear Media", *IEEE MTT-33*, 296-301, 1985.
- [5]. Y. B. Band, D. F. Heller, J. R. Ackerhalt and J. S. Krasinski, "Spectrum of second-harmonic generation for multimode fields", *Physical Review A*, 42, 1515-1521, 1990.
- [6]. Y. B. Band, "Phase-modulation effects on the spectrum of third-harmonic generation", *Physical Review A*, 42, 5530-5536, 1990.
- [7]. Z. A. Tagiev and A. S. Chirkin, "Second Approximation of Dispersion Theory of the Generation of Optical Harmonics", *J. Appl. Spectroscopy* 24, 4, 436-441, 1976.
- [8]. I. Gurwich and D. Censor, "Steady state of electromagnetic waves in weakly nonlinear media", *IEEE Trans. on Magnetics*, 30, 3192-3195, 1994.
- [9]. I. Gurwich and D. Censor, "Propagation of electromagnetic waves with a continuous narrow-band spectrum in weakly nonlinear media", *PIERS*, July 11-15, 1994, Noordwijk, The Netherlands.