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WAVE PACKETS, GROUP VELOCITIES AND RAYS IN LOSSY MEDIA, REVISITED

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ABSTRACT

This study, which also serves as an introduction and a review, was mainly prompted by recent discussions concerning the group velocitiy concept in absorptive media, and the "correctness" of Maxwell's equations, initiated by Harmuth's work. In spite of some derisive comments about the intuitive "textbook" concepts this is exactly the approach used here to develop a consistent definition of the group velocity in absorptive media. It is well-known that the definition of group velocity in absorptive media is not unique. Here we adhere to the hypothesis that in order to describe the transport of energy parcels through real space, the group velocity must be real. Numerical examples are displayed, and theoretical and experimental aspects are briefly discussed. The arguments show that a physically meaningful definition of the group velocity in absorptive media is consistent with the Fermat principle and the special relativistic limitation on the speed of light. Due to space restrictions we present here an abbreviated version.

INTRODUCTION

Harmuth[1-3] and others in his wake[4-11] once more raised the question of the fundamental concepts of wave packets and group velocity in absorptive media. It is not our intention here to criticize in detail the original papers or the ensuing comments, but we do hope that the present review and references will contribute to gain a better and deeper understanding of the basic ideas involved with the extension of the concepts of group velocity, wave packet and ray to the case of absorptive media. In the course of the discussion[1-11], many mathematical tools have been used and their validity for the present problem discussed, but very little has been said about physical concepts and their significance in the related context. When using non-unique concepts such as the group velocity, there are pitfalls which must be obviated, however our impression was that the learned arguments did not help to sharply define these. Our concern, beside the general problem *per se* , is to equip educators with adequate tools for presenting the general concepts to their students. We apologize that our own preferences are emphasized. We use a specific definition for the group velocity, but we realize that our point of view is not the only mathematically consistent model, and not necessarily the only physically meaningful one.

We are not touching on the question of the validity of Maxwell's equations, although it seems that this was one of the motivators of Harmuth and his followers. As an aside, we personally believe that it has not been shown to our satisfaction that Maxwell's equations need modification because of the presence of absorption. A long time ago, the question of solving wave equations (e.g., Maxwell's equations) in the presence of absorptive media has been recognized as a fundamentally difficult one and has been previously dealt with in the literature to some extent, in the context of various branches of physics where absorption is involved. The fact that the present and many other points of view and the pertinent references have not been mentioned in the above cited papers is one of our prime motivators for writing the present paper. It is also worthwhile to emphasize that although some of the above cited work claims to deal with arbitrary time-dependent signals, the solutions always involve integral representations with exponential (in the wide sense, which includes sinousoids with complex arguments) waves in the integrand. Therefore the discussion of transients cannot be divorced from the fundamental question of complex sinousoidal solutions of the wave equation in the presence of absorption. The simple construct of combining at least two waves with neighboring frequencies and propagation vectors, in order to obtain a wave packet train, is the fundamental concept on which the various integral representations are based, for example see Stratton[12]. This is the reason that we propose to review the simple question of narrow band (note, we mean a narrow band of complex frequencies!) signals and the feasibility of constructing wave packets and defining the group velocity in absorptive media. Rather than being unnecessarily polemic, the present approach is constructive: Since it is possible to define the group velocity in various mathematically consistent manners, which at the limit of vanishing small absorption all become identical with the usual "textbook" definition, the problem is to define the group velocity in lossy media in a way that will be theoretically consistent and useful, and at the same time will facilitate experimental verification. To define the group velocity in a manner which mathematically might be consistent, but physically leads to absurdities and inconsistencies (e. g., group velocity exceeding the free space speed of light) is of course possible, but self defeating. Since the definition of group velocity in absorptive media is not unique, all that such an approach can achieve is to show that some specific definitions are useless. It cannot show that such definitions are "wrong" or "right".

The basic difficulty of defining the group velocity in absorptive media stems from the fact that the associated dispersion equation

$$\omega = \omega(\mathbf{k}) \tag{1}$$

or in implicit form $F(\mathbf{k}, \omega) = 0$ is complex. A direct consequence is the fact that the conventional "textbook" definition of the group velocity

$$\mathbf{v}_g = \frac{d\mathbf{x}(t)}{dt} = \frac{\partial \omega(\mathbf{k})}{\partial \mathbf{k}} = \hat{\mathbf{x}}\frac{\partial \omega(\mathbf{k})}{\partial k_x} + \hat{\mathbf{y}}\frac{\partial \omega(\mathbf{k})}{\partial k_y} + \hat{\mathbf{z}}\frac{\partial \omega(\mathbf{k})}{\partial k_z} \tag{2}$$

leads to complex velocities, hence in general complex space-time. The group velocity in absorptive media needs either a conceptual basis, (i. e., an explanation: What are complex velocities and complex space-time coordinates?), or a new definition, which will apply to absorptive media and be consistent with the old definition, i. e., as the wave packet returns to a lossless region, the original definition and real velocities and space-time coordinates will reemerge. The school of thought advocating complex group velocities and space-time coordinates, i.e., the analytical continuation of real quantities into the complex domain, is represented by Jones[13], Budden and Terry[14], Bennett[15], see also Censor, with comments by Suchy[16], and Bennett[17], Wang and Deschamps[18], and Connor and Felsen[19], who also provide ample references to the existing literature. According to this view, there is nothing "wrong" with dealing with complex group velocities. Like other concepts, e.g., frequency, we have to learn to live with its generalization in terms of continuation into the complex regime. Other researchers felt that the concept of a real direction of rays and group velocity, since it involves ideas related to energy conservation, is fundamental and therefore must be retained. For some early work see Barsukow and Ginzburg[20], Storey and Roehner[21], and Terina[22]. In a series of papers Suchy[23-26] investigates the problem of ray propagation in absorbing media. His approach is characterized by constraints that, somewhere along the line, lead to differentiation of nonanalytic expressions. Friedland and Bernstein[27] use as a starting point a nonanalytical "quasi dispersion relation". Since analyticity is closely related to causality, e.g., by the famous Kramers-Kronig relations (which are in fact nothing else but the relations between the real and imaginary parts of an analytic function, e. g., see Kong[28], Landau and Lifshitz[29], and Post[30]). In view of causality requirements, all methods violating analyticity should be considered suspect, in our opinion. For example, it can be shown[31] that nonanalytic group velocities do not satisfy the special-relativistic law of addition of velocities.

Another approach, whose physical basis will be discussed below, has been proposed by Censor and Suchy[32], and further investigated by Censor[33-37], and Censor and Plotkin[38]. The main characteristics of this approach is the constraint that the frequency and wavenumber be chosen such that the group velocity is real. This approach will be further elaborated below.

PHYSICAL CONSIDERATIONS

To avoid being trapped in the webs of the complicated mathematical tools, the simplest model for a wave packet train will be considered first, obtained by "beating" two waves with adjacent frequencies and wavenumbers. The so much scorned "textbook" approach is the mathematically simplest method of introducing the wave packet concept, and in our opinion, contains most of the physics necessary for discussing wave packets and group velocity in absorbing media. By beating two plane waves of adjacent frequency and propagation vector, we obtain a structure in which a "carrier wave" is modulated by an "envelope". If we can identify distinct zeroes between the individual packets in the wave packet train, it can be argued that no energy is transferred from one wavepacket to another across these zeroes. Consequently, the motion of these zero nodes in space can serve as a good method for tagging the group velocity. It is of course known that in the presence of strong dispersion the wave packet structure is rapidly obliterated, thus rendering the whole idea of group velocity meaningless. Therefore it must be born in mind that any discussion of this kind, the present one included, assumes moderate dispersion and narrow band signals.

Consider the "textbook" problem of adding two plane waves having real $\mathbf{k}$, ω *in a lossless medium:*

$$\begin{aligned} \phi &= \mathrm{Re}\{e^{i(\mathbf{k}+\Delta\mathbf{k})\cdot\mathbf{x} - i(\omega+\Delta\omega)t} - e^{i(\mathbf{k}-\Delta\mathbf{k})\cdot\mathbf{x} - i(\omega-\Delta\omega)t}\} \\ &= 2\mathrm{Re}\{ie^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}\sin(\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t)\} \\ &= -2\sin(\mathbf{k}\cdot\mathbf{x} - \omega t)\sin(\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t) \end{aligned} \tag{3}$$

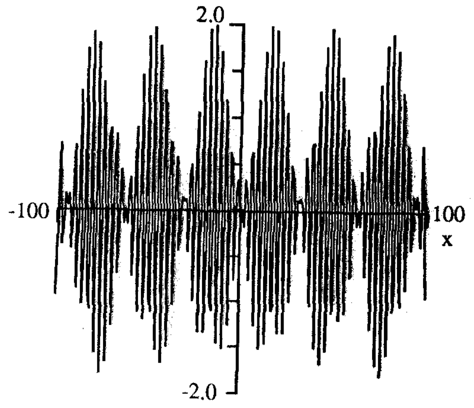

Fig. 1: Wave packet train produced by beating two plane waves in a lossless medium. The figure shows the wave at $t = 0$, in the range $x=-100$ to 100, with $\mathbf{k} = 2$ and $\Delta\mathbf{k} = 0.1$.

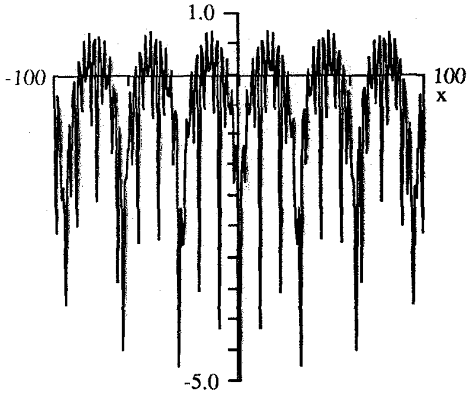

Fig. 2: Natural logarithm of the ordinate of the graph in Fig. 1 emphasizes the zeroes of the beat wavepacket train.

The choice of the minus sign between the two waves in the first line of (3) is arbitrary, its relevance to one of our examples will be explained below. Approximating $\Delta\omega = \frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}}\cdot\Delta\mathbf{k}$ and neglecting higher order derivatives, which only makes sense for media in which we have moderate dispersion, and substituting in (3), then defining the argument of the envelope $\sin(\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t)$ in (3) by means of $\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t = \text{constant}$, it is clear that the wave packet moves at the group velocity (2) and can be tracked by any phase of the envelope, in particular by following its zeroes. See fig. 1. The graph is drawn from a sequence of discrete points and therefore the zeroes of the envelope are not clearly seen. The nature of the zeroes is emphasized in Fig. 2, in which the amplitude is the natural logarithm of the graph in Fig. 1. The situation becomes more complicated and we immediately lose the insight provided by the wave packet train when a lossy medium is involved. Assuming that the excitation is still time harmonic, we choose real ω and real $\Delta\omega$. Consequently $\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}}\cdot\Delta\mathbf{k}$ is real too, but $\mathbf{k}$, $\Delta\mathbf{k}$ and $\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}}$ are complex, in general. For complex arguments the extension of (3) becomes

$$\phi = \mathrm{Re}\{e^{i(\mathbf{k}+\Delta\mathbf{k})\cdot\mathbf{x} - i(\omega+\Delta\omega)t} - e^{i(\mathbf{k}-\Delta\mathbf{k})\cdot\mathbf{x} - i(\omega-\Delta\omega)t}\} = \mathrm{Re}\{ e^{iA} - e^{iB}\}$$
$$= e^{-\mathrm{Im}A}\cos(\mathrm{Re}A) - e^{-\mathrm{Im}B}\cos(\mathrm{Re}B) \qquad (4)$$

and will be considered, here as before, for real $\mathbf{x}$, t. The main feature of (4) is the fact that for real frequencies the envelope does not have real zeroes. As a consequence, a typical "beat" between two waves with adjacent real frequencies, and due to the losses complex $\mathbf{k}$, $\Delta\mathbf{k}$, , loses the typical wavepacket train structure of Figs. 1, 2. For small losses the beat appears as if the zeroes are still there, as seen in Fig. 3. This is so because in the complex plan the complex zeroes are not far from the real axis. However, we wish to learn about the effect of losses in general, hence this approximation is trivial. For higher losses as in Fig. 4 it becomes clear that as the complex zeroes are farther from the real axis, the structure of the distinct pulses in the wavepacket train becomes obliterated, and the tracking of zeroes through space becomes impossible. Inasmuch as the exponential decay has a strong effect, it also becomes ambiguous whether local maxima are indeed maxima of the wavepackets or that the wavepacket maxima have been displaced due to the effect of the exponential attenuation. Therefore the whole idea of defining a real group velocity under these circumstances becomes untenable. By inspecting fig. 4 it becomes clear why for small losses people tend to adhere to the old model of using real frequencies, and what they have to pay for that: For small attenuation and real frequencies, about $\mathbf{x}=0$ (or more generally, in the vicinity of the reference position) the beat can "more or less" be recognized and the positions of the minima of the envelope can be identified. However, to search for wave packets and group velocities in these circumstances is *conceptually flawed*, because it is seen that away from the reference distance the distinct wave packets become obliterated. For the general case of an integral over plane waves, this means that real frequencies cannot serve to construct a proper wave packet in absorptive media, because the mechanism for destructive interference fails outside the zone where the wave packet is located! This concept of wave interference which enhances the field in a localized region of space and annihilates it elsewhere is the basis for the ideas of wavepacket and group velocity. In Fig. 5 the logarithm of the amplitude is shown, demonstrating the disappearance of the zeroes of the envelope. The downward spikes are points where the discrete sampling of (4) was at points close to zero passes of the oscillating signal. The large downward spike at $\mathbf{x}=0$ indicates the only real zero of the envelope. This remark will be amplified below.

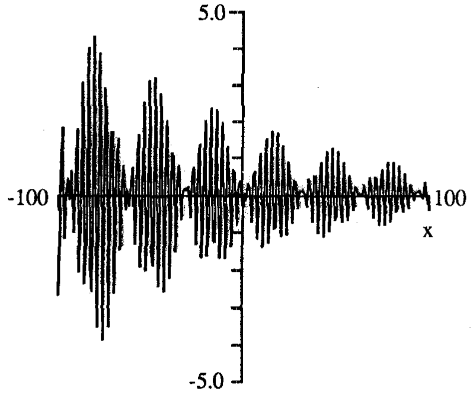

Fig. 3: Graph of Eqn. (4) for weak losses. The figure shows the wave at $t = 0$, in the range $x=-100$ to 100, with $\mathrm{Re}\mathbf{k} = 2$, $\mathrm{Im}\mathbf{k} = 0.01$, $\mathrm{Re}\Delta\mathbf{k} = 0.1$, $\mathrm{Im}\Delta\mathbf{k} = 0.0005$. The exponential decay of the envelope is evident, but it appears that the zeroes in the wavepacket train can still be identified.

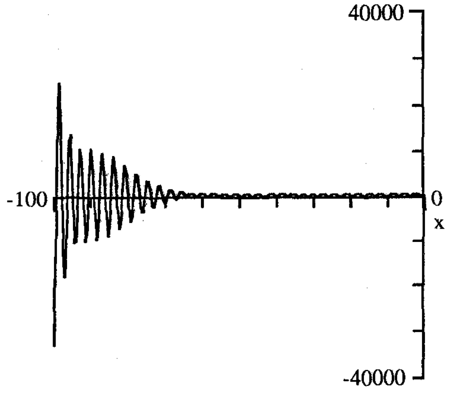

Fig. 4: Graph of Eqn. (4) for moderate losses. The figure shows the wave at $t=0$, in the range $x= 0$ to 100, with $\mathrm{Re}\mathbf{k} = 2$, $\mathrm{Im}\mathbf{k} = 0.1$, $\mathrm{Re}\Delta\mathbf{k} = 0.1$, $\mathrm{Im}\Delta\mathbf{k} = 0.005$. In addition to the exponential decay , the individual pulses of the wavepacket train are obliterated. The slight upward displacement from the $\mathbf{x}$-axis aids to see the behaviour for low amplitudes.

As the attenuation becomes stronger, even the individual oscillations are strongly attenuated and deformed, as seen in fig. 6. No individual wavepackets can be identified under such circumstances, and the concept of group velocity in its classical form loses its meaning.

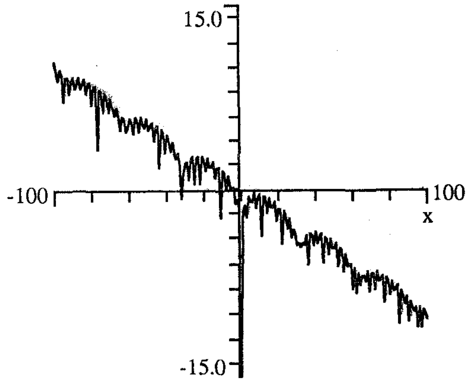

Fig. 5: Natural logarithm of the ordinate of the graph in Fig. 4, here x=-100 to 100. The graph shows how the zeroes between individual wavepackets are obliterated. There exists a real zero at x=0 which is discussed in the text.

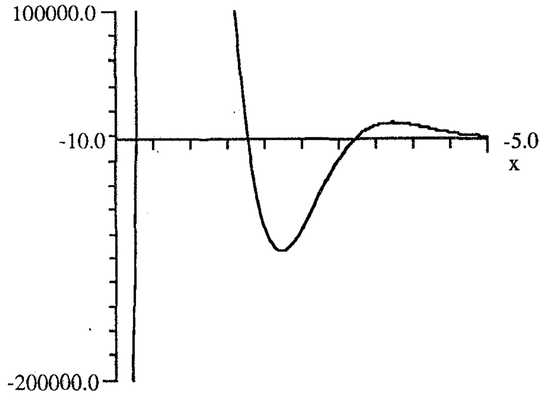

Fig. 6: Graph of Eqn. (4) for high losses. The figure shows the wave at t=0, in the range x=-10 to -5, with $\mathrm{Re}\mathbf{k} = 2$, $\mathrm{Im}\mathbf{k} = 1$, $\mathrm{Re}\Delta\mathbf{k} = 0.1$, $\mathrm{Im}\Delta\mathbf{k} = 0.5$. Even the individual oscillations are deformed. Individual wave pakets and the associated group velocity cannot be measured under such circumstances.

Does this mean that the concepts of wavepacket and group velocity are not applicable to lossy media? We believe they still are, provided proper definitions are used. This conclusion stems from the realization that the definition of the wavepacket and group velocity concepts in lossy media do not have to be based on signals having real frequencies. Even if the source emits a burst, which in space is a wave packet, and at the source this burst can be represented in terms of a superposition of real frequency waves, the changes occurring to this wavepacket as it enters and propagates through a lossy environment might distort it in such a way that it appears as a superposition of complex frequency waves. What we propose to do in the next section is to show that real group velocities can be defined in a meaningful manner in lossy media, by defining wavepackets in terms of complex frequencies and propagation vectors.

REAL GROUP VELOCITIES IN LOSSY MEDIA

In order to investigate the validity of the group velocity concept in absorptive systems, Censor[34] considered the extreme case where the attenuation is predominant, to the extent that oscillations are completely excluded by the system. Or, in terms of the time derivatives appearing in the differential equations, the second and higher order time derivatives vanish, and only the first time derivative exists. A physical example for such a case is provided by the well-known heat conduction or diffusion equation[39]. The result is similar to Fig. 6. For more detail see the full length version.

The general idea can be expressed in the following way. Given the the dispersion equation (1) or in general $F(\mathbf{k}, \omega) = 0$, we form the group velocity according to (2) (which of course requires *a-priori* that the multivariate $F(\mathbf{k}, \omega) = 0$ be analytic in all its variables) and impose the constraint

$$\mathrm{Im}\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}} = 0 \tag{5}$$

which describes a surface in a six dimensional space $\mathrm{Re}\mathbf{k}$, $\mathrm{Im}\mathbf{k}$. We thus have to choose $\mathbf{k}, \omega$ which satisfy $F(\mathbf{k}, \omega) = 0$ and (5) simultaneously.The analyticity of the multivariate function $\frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}}$ means that in the limiting process of defining the derivatives, the choice of the increments $\Delta\mathbf{k}$ in the complex regime is arbitrary, meaning that $\Delta\mathbf{k}$ can be chosen as real, no matter what the choice of $\mathbf{k}, \omega$ is. By also choosing real $\mathbf{x}$, t, as stipulated all along, we thus ensure that the argument of $\sin(\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t)$ $= \sin\left(\Delta\mathbf{k}\cdot\left[\mathbf{x} - \frac{\partial\omega(\mathbf{k})}{\partial\mathbf{k}}t\right]\right)$ be real, hence $\mathrm{Im}[\sin(\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t)] = 0$ identically. It follows that for this case (4) becomes

$$\phi = -2[\mathrm{Im}\, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}]\sin(\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t) \tag{6}$$

Consequently the zeroes of the envelope in (6) exist, and are the zeroes of $\sin(\Delta\mathbf{k}\cdot\mathbf{x} - \Delta\omega t)$, which means that we have again the structure of distinct wave packets separated by zero amplitude nodes, moving through real space with a real group velocity $\mathbf{v}_g$ on real trajectories prescribed by $\mathbf{x} = \mathbf{v}_g t$ as in the lossless case. However, we have now complex $\mathbf{k}, \omega$ whose presence in the carrier wave exponentially modulates the wave packet train in space and time. Thus we have demonstrated that a lossy medium can support wave packets propagating with a real group velocity. In absorptive media, in order to have real $\mathbf{v}_g$, complex values must be chosen for $\mathbf{k}, \omega$. The feasibility of such a program will be demonstrated below on a simple example of a homogeneous, cold, unmagnetized, collisional plasma model, such as is hypothesized for the ionosphere. Note that in the above given example of the limiting case of the diffusion equation, the increments of $\mathbf{k}, \omega$ have been chosen such that a sinh function was derived. This was a matter of convenience. Another important aspect which must be clarified is the following: Inasmuch as we start in (4) with only two plane waves which are infinite in space and time, the question of launching a wave packet into a lossy medium cannot be discussed in this context, and must considered separately.

A more realistic situation, which takes into account the launching aspect, has been analysed before[33]. Following Connor and Felsen[19], a Gaussian plane pulse injected into an absorbing medium is considered. At the boundary x = 0 where the wave is launched a signal as a function of time is given:

$$\phi(x=0, t) = e^{[-i\omega_0 t - (t/2\alpha)^2]} \tag{7}$$

where α, ω_0 are positive real constants. For $x > 0$ we use the Fourier transform of (7) in order to extend (7) to arbitrary x and thus construct a wave equation solution

$$\phi(x, t) = \frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-(\omega-\omega_0)^2 + ik(\omega)x - i\omega t} \tag{8}$$

where $k = k(\omega)$ is the dispersion equation (in a one dimensional situation where both k and ω are scalars, it is possible to use $k(\omega)$ instead of $\omega(k)$, assuming the function can be inverted) and the prime indicates the integration variable. We now approximate $k(\omega) = k(\omega_s) + \frac{dk}{d\omega_s}(\omega-\omega_s)$, $\omega_s = \omega_0 + i\mu$ where ω_0, μ are real, and ω_s is chosen such that $dk/d\omega_s = 1/v_g$ is real. substituting in (8) we obtain the wave packet structure

$$\phi(x, t) = e^{ik_s x - i\omega_s t}\frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-\alpha^2(\omega-\omega_0)^2 - i(\omega-\omega_s)\theta}$$

$$\theta = t - \frac{dk}{d\omega_s}x, \quad k_s = k(\omega_s) \tag{9}$$

clearly displaying the carrier and the envelope factors. The envelope, i.e., the integral in (9) is constant on $\theta = \mathrm{constant}$. For the present case the integration can be performed, yielding in explicit form

$$\phi(x, t) = e^{ik_s x - i\omega_s t}\, e^{-\mu\theta - (\theta/2\alpha)^2} \tag{10}$$

The propagation of such a wave packet will be demonstrated for a concrete physical system of an unmagnetized, cold, collisional plasma. This will serve to demonstrate both how signals can be launched into an abscrptive medium and the choice of complex $\mathbf{k}, \omega$ which guarantee a real group velocity. The dispersion equation for this case is a special case of the well-known Appleton-Hartree equation which has been extensively applied to ionospheric propagation[40, 41], and for the present case reduces to

$$F(\mathbf{k},\omega) = (i\omega - \nu)k^2 - \omega^2(i\omega - \nu)\varepsilon_0\mu_0 + i\omega\omega_p^2\varepsilon_0\mu_0 = 0 \tag{11}$$

where ν is the collision frequency. Essentially $F(\mathbf{k}, \omega) = 0$ is obtained from a four-dimensional space-time Fourier transformation, therefore (11) corresponds to the wave equation

$$\left[\left(\frac{\partial}{\partial t}+\nu\right)\frac{\partial}{\partial x}\cdot\frac{\partial}{\partial x}-\varepsilon_0\mu_0\left(\frac{\partial}{\partial t}+\nu\right)\frac{\partial^2}{\partial t^2}-\varepsilon_0\mu_0\omega_p^2\frac{\partial}{\partial t}\right]\phi(x,t)=0 \qquad (12)$$

which clearly demonstrates the various orders of the time derivatives. In the present case the time derivative appears to the orders 1, 2, 3. In order to implement the condition (5) explicitly, first solve (11) for k then differentiate with respect to ω, yielding $dk/d\omega$. Inasmuch as we are dealing with a one-dimensional problem, the inverse yields the group velocity, i.e., $v_g = d\omega/dk$. The realness of v_g is prescribed by

$$\operatorname{Im}\left\{\frac{d\omega}{dk}\right\}=\operatorname{Im}\left\{\frac{2\sqrt{(i\omega-\nu)^3(i\omega^3-i\omega\omega_p^2-\omega^2\nu}}{\sqrt{\varepsilon_0\mu_0}\,(2\omega\nu^2-2\omega^3-i4\omega^2\nu-i\omega_p^2\nu)}\right\}=0 \qquad (13)$$

which in turn prescribes a relation between $\omega_r = \omega_0$ and $\omega_i = \mu$. The expression (13) is very complicated to evaluate, details are given in Appendix A of the full text version. In terms of expressions defined in Appendix A, (13) becomes a transcendental equation prescribing allowed values of $\omega = \omega_r + i\omega_i$. The corresponding $k = k_r + k_i$ are then computed. Computed results are shown in the full text version.

We now return to the Gaussian pulse (10). Of course, since the Gaussian envelope does not possess zeroes in the amplitude, we cannot follow zeroes like we did above in the wave packet train case. In practical cases the present situation is the common case. In spite of that, we are able to assign a real group velocity to wave packets propagating in absorbing media, and it is obvious that as absorption vanishes, the expression for the group velocity coincides with the conventional one for lossless media. Computed values are chosen for $\omega_r = \omega_0$ and $\omega_i = \mu$ and the corresponding real and imaginary values for k_g. Substituting in (10) and taking the real part, the function describing the wave packet becomes

$$\phi(x,t) = \exp\left[\left(\frac{\omega_i}{v_g}-k_i\right)x-\left(\frac{t-x/v_g}{2\alpha}\right)^2\right]\cos(k_r x-\omega_r t) \qquad (14)$$

For x=constant, e.g., x=0, it is clear from (14) that we have a carrier modulated by the Gaussian envelope. As x increases the wave packet propagates into the absorbing medium. If we follow it at the group velocity, i.e., according to $t=x/v_g$ then the term involving α in (14) remains constant. As the wavepacket propagates into the medium it should be attenuated according to

$$\exp\left[\left(\frac{\omega_i}{v_g}-k_i\right)x\right] = \exp\left[\omega_i\left(\frac{1}{v_g}-\frac{1}{v_p}\right)x\right] \qquad (15)$$

Clearly the sign of the exponent (15) will be negative for positive x and negative ω_i if $v_g < v_p$, where v_p is the phase velocity. This is the case in normal dispersive media. In Figs. 12-14 typical wavepackets are plotted for various times as a function of x. Therefore the pulse is attenuated as it propagates deeper into the medium. It is also interesting to note that the peak of the amplitude is not at $x=v_g t$ but for smaller x, which provides a larger amplitude due to smaller attenuation in the exponent in (15). This is an interesting result that could not be anticipated by using other models. Typical results are displayed in Figs. 7, 8.

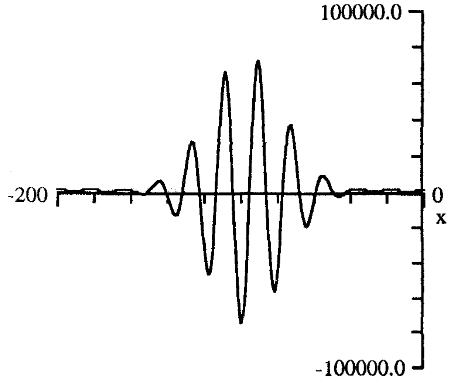

Fig. 7: A typical wavepacket in an absortptive medium, parameters are chosen arbitrarily: $\omega_r=1$, $\varepsilon_0\mu_0=1$, $\omega_p^2=0.8$, $\alpha=30$, $t=0$.

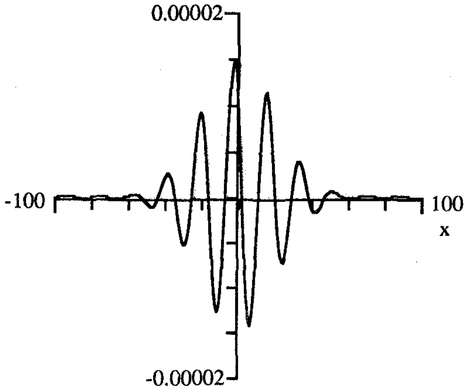

Fig. 8: As in Fig. 12, but t=200.

Thus we have demonstrated the feasibility of describing propagation of wavepackets in absorptive media, and although the shape of the pulse changes in space and time, the formulation of a real group velocity is possible. In the next chapter these ideas are extended to the case of inhomogeneous absorptive media.

RAY PROPAGATION IN ABSORPTIVE MEDIA

One of the reasons for developing the concepts of wave packet and group velocity was the need to discuss wave propagation in inhomogeneous and sometimes also time varying media, for which we are unable to provide analytic solutions to the field (e.g., Maxwell's) equations. It is therefore interesting to note in the present context that ray propagation in slowly varying absorbing media have been discussed before. Studies cited above address this problem and provide citations to the relevant literature. In particular, we would like to mention the formalism based on (10), which originated with Censor and Suchy[32] and was further investigated by Censor[31-38], for simple introductions see Censor[42, 43]. For detail see the full length version.

EXPERIMENTAL CONSIDERATIONS

This subject is discussed in detail in the full length version.

THEORETICAL NOTES

The question of analyticity of $F(\mathbf{k}, \omega) = 0$ and v_g and the way they enter our problem is discussed in the full length version. The questions of the relativistic transformation of the group velocity and the restriction that it does not exceed c, the free space speed, are discussed too.

DISCUSSION AND CONCLUDING REMARKS

The concept of group velocity is approximate, based on the assumption that we deal with a medium possessing small dispersion, such that the dispersion equation (1) can be satisfactorily approximated by retaining the first derivative only anf discarding higher terms. This assumption fails for systems where the dispersion is not small, and the signal is not sufficiently narrow band. In spite of the disadvantages, the ideas of wave packets propagating on ray trajectories at the group velocity is invaluable for dealing with inhomogeneous, time-varying and moderately dispersive media, because exact analytical solutions to the field equations are not available. It is therefore inconceivable that such an elusive concept, which becomes exact only when dispersion vanishes, hence when it is not necessary, can be used to test the validity of first principles, i.e., the field equations themselves. In lossy media where (1) is complex, the problem is further complicated by the fact that the conventional definition (2) of the group velocity, yields complex values, implying complex space-time. At this point it is also becoming obvious that the extension of the concept to absorptive media is not unique, which again contributes to the confusion. Certain approaches to defining and measuring the group velocity, whatever the researcher has in mind when using this term, lead to outright absurdities, like group velocities exceeding the speed of light in vacuum. Rephrasing the last statement, we should say that the use of the term group velocity, which has been devised to describe the propagation of energy parcels through space, should be consistent with pertinent results prescribed by special relativity. The approach should therefore be pragmatical and contribute to our ability to analyze physical systems of interest.

With this in mind, the present study has shown that real group velocity can be defined for lossy media. The concept has been theoretically discussed and its physical implications have been explained, and numerical examples provided to illustrate these arguments. The relevance to the analytic properties of the dispersion equation has been mentioned. The extension of Hamilton's ray equations for absorptive media was given. This formalism has been numerically used for ray tracing in an absorptive ionosphere.

Due to the non-uniqueness of the problem, more experimental data is required, in order to determine the most suitable definition of group velocity for absorptive media. A possible experiment has been described which might contribute towards a better understanding of the subject.

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APPENDIX A: EVALUATION OF EQUATION (13)

The details of this section are presented in the full length manuscript.