IEEE 26th Annual Northeast Bioengineering Conference, Storrs, University of Connecticut, April 9-10, 2000.

First-Order Material Effects In Gyromagnetic Systems

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1] method used for ocular glucose polarimetry is based on the Faraday rotation phenomenon encountered in gyromagnetic media. In liquids and solids the ambient isotropic relative permittivity is large compared to the gyromagnetic effects, suggesting that a first order perturbation scheme be used to analyze the problem. The general theory for such a scheme is presented here*.

I. INTRODUCTION

In the time harmonic domain with the harmonic factor given by $\exp(-i\omega t)$, and the simple relations $\mathbf{D}=\varepsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$, where the constitutive parameters are scalars, the Maxwell equations [2] are given by

$$\begin{aligned}\partial_{\mathbf{r}}\times\mathbf{H}&=-i\omega\varepsilon\mathbf{E}+\mathbf{j}_e\\ \partial_{\mathbf{r}}\times\mathbf{E}&=i\omega\mu\mathbf{H}-\mathbf{j}_m\\ \partial_{\mathbf{r}}\cdot\mathbf{E}&=\rho_e/\varepsilon\\ \partial_{\mathbf{r}}\cdot\mathbf{H}&=\rho_m/\mu\end{aligned}\qquad(1)$$

where the indices e, m refer to electric, magnetic sources, respectively. The magnetic sources are added for completeness, although they do not have an independent physical meaning. However they provide a mathematical tool for dealing with problems, e.g., the perturbation scheme used here. The system of equations (1) is recast in the form:

$$\begin{array}{l|l}\partial_{\mathbf{r}}\times\mathbf{E}_e=i\omega\mu\mathbf{H}_e & \partial_{\mathbf{r}}\times\mathbf{E}_m=i\omega\mu\mathbf{H}_m-\mathbf{j}_m\\ \partial_{\mathbf{r}}\times\mathbf{H}_e=-i\omega\varepsilon\mathbf{E}_e+\mathbf{j}_e & \partial_{\mathbf{r}}\times\mathbf{H}_m=-i\omega\varepsilon\mathbf{E}_m\\ \partial_{\mathbf{r}}\cdot\mathbf{E}_e=\rho_e/\varepsilon & \partial_{\mathbf{r}}\cdot\mathbf{E}_m=0\\ \partial_{\mathbf{r}}\cdot\mathbf{H}_e=0 & \partial_{\mathbf{r}}\cdot\mathbf{H}_m=\rho_m/\mu\end{array}\qquad(2)$$

where $\mathbf{E}=\mathbf{E}_e+\mathbf{E}_m$ etc., and the indices on the fields indicate whether they are e, or m, induced. Obviously the summation of the two sets of equations (2) yields back the original Maxwell equations (1). The two systems in (2) possess similitude properties

$$\mathbf{j}_e\Leftrightarrow-\mathbf{j}_m,\rho_e\Leftrightarrow-\rho_m,\mathbf{E}_e\Leftrightarrow\mathbf{H}_m,\mathbf{H}_e\Leftrightarrow\mathbf{E}_m,\varepsilon=-\mu\qquad(3)$$

Accordingly, by substituting into the left set in (2) the corresponding fields indicated by (3), the right hand side of (2) is obtained, and *vice-versa*.

Applying a divergence operator to the two vector equations of each set in (2) leads to

$$\partial_{\mathbf{r}}\cdot\mathbf{j}_{e,m}-i\omega\rho_{e,m}=0\qquad(4)$$

written together for the e, m, indices, called the equation of continuity, or conservation of charge.

The general solution for the fields in terms of the source current is given, e.g., in [2]

$$\begin{aligned}\mathbf{E}(\mathbf{r})&=\int_{V(\mathbf{r}')}dV(\mathbf{r}')\{\tilde{\Gamma}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_e(\mathbf{r}')i\omega\mu-\partial_{\mathbf{r}'}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_m(\mathbf{r}')\}\\ \mathbf{H}(\mathbf{r})&=\int_{V(\mathbf{r}')}dV(\mathbf{r}')\{\tilde{\Gamma}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_m(\mathbf{r}')i\omega\varepsilon+\partial_{\mathbf{r}'}\times\tilde{\Gamma}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}_e(\mathbf{r}')\}\end{aligned}\qquad(5)$$

where the two equations are similar subject to the transformation (3). In (5)

$$\begin{aligned}&\tilde{\Gamma}(\mathbf{r},\mathbf{r}')=(\tilde{\mathbf{I}}+k^{-2}\partial_{\mathbf{r}}\partial_{\mathbf{r}})G(\mathbf{r},\mathbf{r}'),k^2=\omega^2\mu\varepsilon\\ &G(\mathbf{r},\mathbf{r}')=e^{ik|\mathbf{r}-\mathbf{r}'|}/4\pi|\mathbf{r}-\mathbf{r}'|\end{aligned}\qquad(6)$$

where $G,\tilde{\Gamma}$ are the scalar, and its corresponding dyadic, respectively, free space Green functions

II. THE PERTURBATION METHOD

In general, we consider general bi-anisotropic media whose constitutive relations can be included in the form

$$\begin{aligned}\mathbf{D}&=\varepsilon\mathbf{E}+\tilde{\mathbf{a}}\cdot\mathbf{E}+\tilde{\mathbf{b}}\cdot\mathbf{H}\\ \mathbf{B}&=\mu\mathbf{H}+\tilde{\mathbf{c}}\cdot\mathbf{H}+\tilde{\mathbf{d}}\cdot\mathbf{E}\end{aligned}\qquad(7)$$

where ε,μ are scalars, and $\tilde{\mathbf{a}},\tilde{\mathbf{b}},\tilde{\mathbf{c}},\tilde{\mathbf{d}}$, are the pertinent dyadics. The only limitation on the dyadics is that they can be considered as perturbing the simple media.

In the perturbation scheme we understand all the fields to have a zero order and first order components, e.g. $\mathbf{E}=\mathbf{E}_0+\mathbf{E}_1$ etc. Substituting the fields into the constitutive relations, and separating zero order and first order terms, we obtain,

$$\mathbf{D}_0=\varepsilon\mathbf{E}_0,\mathbf{B}_0=\mu\mathbf{H}_0\qquad(8)$$

$$\begin{aligned}\mathbf{D}_1&=\varepsilon\mathbf{E}_1+\tilde{\mathbf{a}}\cdot\mathbf{E}_0+\tilde{\mathbf{b}}\cdot\mathbf{H}_0\\ \mathbf{B}_1&=\mu\mathbf{H}_1+\tilde{\mathbf{c}}\cdot\mathbf{H}_0+\tilde{\mathbf{d}}\cdot\mathbf{E}_0\end{aligned}\qquad(9)$$

* The present study was motivated by a joint research on ocular polarimetric glucometry initiated by Torsana Diabetes Diagnostics A/S, Skodsborg, Denmark, and the University of Connecticut.

here terms of the order $\tilde{\mathbf{a}} \cdot \mathbf{E}_1$ etc. are neglected provided the dyadics are small. Substituting $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$ etc. into (2) and separating zero order and first order terms, we obtain for the zero order fields (2) itself, with index zero attached to all field and source variables. The solution for the zero order fields is achieved by performing the integrations (5), with the dyadic Green function (6) and with index zero attached to all field and source variables. The solutions for specific problems depend on the aforehand provided source fields, and in comparison to the full fledged unperturbed solution are simpler to derive.

The first order perturbation scheme prescribes

$$\begin{aligned} \partial_{\mathbf{r}} \times \mathbf{H}_1 &= -i\omega\varepsilon\mathbf{E}_1 - i\omega(\tilde{\mathbf{a}} \cdot \mathbf{E}_0 + \tilde{\mathbf{b}} \cdot \mathbf{H}_0) = -i\omega\varepsilon\mathbf{E}_1 + \mathbf{j}_{1,e} \\ \partial_{\mathbf{r}} \times \mathbf{E}_1 &= i\omega\mu\mathbf{H}_1 + i\omega(\tilde{\mathbf{c}} \cdot \mathbf{H}_0 + \tilde{\mathbf{d}} \cdot \mathbf{E}_0) = i\omega\mu\mathbf{H}_1 - \mathbf{j}_{1,m} \\ \partial_{\mathbf{r}} \cdot \mathbf{D}_1 &= \varepsilon\partial_{\mathbf{r}} \cdot \mathbf{E}_1 = -\partial_{\mathbf{r}} \cdot (\tilde{\mathbf{a}} \cdot \mathbf{E}_0 + \tilde{\mathbf{b}} \cdot \mathbf{H}_0) = \rho_{1,e} \\ \partial_{\mathbf{r}} \cdot \mathbf{B}_1 &= \mu\partial_{\mathbf{r}} \cdot \mathbf{H}_1 = -\partial_{\mathbf{r}} \cdot (\tilde{\mathbf{c}} \cdot \mathbf{H}_0 + \tilde{\mathbf{d}} \cdot \mathbf{E}_0) = \rho_{1,m} \end{aligned} \qquad (10)$$

where on the right-hand side we have identified the first order current and charge sources, in terms of the supposedly already known fields obtained by solving for the zero order terms, using (6). Thus the new quantities $\mathbf{j}_{1,e}, \mathbf{j}_{1,m}, \rho_{1,e}, \rho_{1,m}$ are not additional unknowns, but rather known functions. As such, they allow us to consider them as the source (inhomogeneous) terms of the differential equations (10).

All former zero order expressions follow through for the first order terms, e.g., (4), and therefore we are still dealing formally with an isotropic medium, and the solution for the first order perturbation fields is given by the analog of (5), i.e.,

$$\begin{aligned} \mathbf{E}_1(\mathbf{r}) &= \int_{V(\mathbf{r}')} dV(\mathbf{r}')\left\{\tilde{\Gamma}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,e}(\mathbf{r}')i\omega\mu - \partial_{\mathbf{r}'} \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,m}(\mathbf{r}')\right\} \\ \mathbf{H}_1(\mathbf{r}) &= \int_{V(\mathbf{r}')} dV(\mathbf{r}')\left\{\tilde{\Gamma}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,m}(\mathbf{r}')i\omega\varepsilon + \partial_{\mathbf{r}'} \times \tilde{\Gamma}(\mathbf{r},\mathbf{r}') \cdot \mathbf{j}_{1,e}(\mathbf{r}')\right\} \end{aligned} \qquad (11)$$

The above results are complicated by a lot of detail, but nevertheless provide explicit formulas for the derivation of the first order anisotropic material effects from the zero order solution. In order to compact the notation, some shorthand conventions will be introduced. We introduce the solution in the form of integral dyadic operators acting on the zero-order solutions

$$\begin{aligned} \mathbf{E}_1 &= \left(\int \tilde{\mathscr{A}}\right) \cdot \mathbf{E}_0 + \left(\int \tilde{\mathscr{B}}\right) \cdot \mathbf{H}_0 \\ \mathbf{H}_1 &= \left(\int \tilde{\mathscr{C}}\right) \cdot \mathbf{H}_0 + \left(\int \tilde{\mathscr{D}}\right) \cdot \mathbf{E}_0 \end{aligned} \qquad (12)$$

In (12) the fields on the left, $\mathbf{E} = \mathbf{E}(\mathbf{r}), \mathbf{H} = \mathbf{H}(\mathbf{r})$, are functions of the location vector $\mathbf{r}$, while on the right we have $\mathbf{E}_0 = \mathbf{E}_0(\mathbf{r}'), \mathbf{H}_0 = \mathbf{H}_0(\mathbf{r}')$, which are expressed in terms of the integration variables $\mathbf{r}'$, and thus operated upon within the integral by the dyadic operators defined by boldface script characters. The integral symbol stands for $\int = \int_{V(\mathbf{r}')} dV(\mathbf{r}')$ as in (5) and (11). The dyadic operators are given by

$$\begin{aligned} \tilde{\mathscr{A}} &= \omega^2\mu\tilde{\Gamma} \cdot \tilde{\mathbf{a}} + i\omega\partial_{\mathbf{r}'} \times \tilde{\Gamma} \cdot \tilde{\mathbf{d}} \\ \tilde{\mathscr{B}} &= \omega^2\mu\tilde{\Gamma} \cdot \tilde{\mathbf{b}} + i\omega\partial_{\mathbf{r}'} \times \tilde{\Gamma} \cdot \tilde{\mathbf{c}} \\ \tilde{\mathscr{C}} &= \omega^2\varepsilon\tilde{\Gamma} \cdot \tilde{\mathbf{c}} - i\omega\partial_{\mathbf{r}'} \times \tilde{\Gamma} \cdot \tilde{\mathbf{b}} \\ \tilde{\mathscr{D}} &= \omega^2\varepsilon\tilde{\Gamma} \cdot \tilde{\mathbf{d}} - i\omega\partial_{\mathbf{r}'} \times \tilde{\Gamma} \cdot \tilde{\mathbf{a}} \end{aligned} \qquad (13)$$

We have accomplished our goal: The first-order solution of the field problem in the presence of an arbitrary anisotropicity inclusion is given by the solution of the limiting case simple medium, and the first order correction is accomplished by operations on these solutions.

III. Gyromagnetic Media

Gyromagnetic media are characterized by a preferred direction, introduced by the external magnetic field, rendering the medium anisotropic. Here the gyromagnetic effect is considered to be small, and therefore the above given perturbation method will be applied.

For a representative text with numerous references to the literature see Kelso [3]. In essence, we are interested here in polarization effects produced by electric fields, in the presence of static or *quasi* static magnetic fields. In liquids and solids there are quite a few polarization mechanisms, e.g., dipole or orientational, ionic or molecular, electronic, polarizations. In addition we have ionized gas polarization in plasma media, for which an extensive analysis exists. Our main interest here concerns the previous categories involving solid and fluid media. Such materials are exploited in electro-optical devices required for applications where polarization measurements are performed. It is therefore necessary to define a model which will account for the gyromagnetic effects in such materials.

We start with the analysis of gyromagnetic effects in ionized gases [3] and subsequently derive expressions for similar effects in fluids and solids. The details will be published elsewhere.

References

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