EE 376A/Stat 376A
Information Theory
Prof. T. Cover

Handout #30
Monday, March 14, 2005

Final Examination

1. *(20 points)* **Cookies.**
Let

$$V_n = \prod_{i=1}^{n} X_i,$$

where X_i are i.i.d.

$$X_i = \begin{cases} 1/8, & \text{probability } 1/2, \\ 1/2, & \text{probability } 1/2. \end{cases}$$

Presumably, X_i is the fraction remaining after a single mouse bite.

(a) Let

$$V_n' = \alpha^n.$$

Find the value of α such that V_n and V_n' decrease at the same rate.

For parts (b) and (c), we mix V_n and V_n' as follows. Let

$$Y_i = \lambda\alpha + (1-\lambda)X_i,$$

where $\lambda \in (0,1)$. Let

$$V_n'' = \prod_{i=1}^{n} Y_i.$$

(b) Is the growth rate of V_n'' larger or smaller than $\log\alpha$?

(c) What is the growth rate of V_n'' for $\lambda = 1/2$?

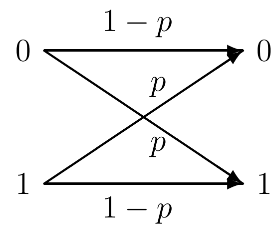
2. *(20 points)* **Huffman code.**
Find the binary Huffman encoding for

$$X \sim \mathbf{p} = \left(\frac{19}{40}, \frac{8}{40}, \frac{3}{40}, \frac{3}{40}, \frac{3}{40}, \frac{2}{40}, \frac{2}{40} \right).$$

3. *(20 points)* **Good codes.**
Which of the following codes are possible Huffman codes?

(a) $\{0,00,01\}$
(b) $\{0,10,11\}$
(c) $\{0,10\}$

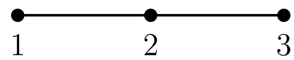
4. *(20 points)* **Errors and erasures.**
Consider a binary symmetric channel (BSC) with crossover probability p.



A helpful genie who knows the locations of all bit flips offers to convert flipped bits into erasures. In other words, the genie can transform the BSC into a binary erasure channel. Would you use his power? Be specific.

5. *(40 points)* **Random walks.**
Consider the following graph with three nodes:

$\{X_i\}$



(a) What is the entropy rate $H(\mathcal{X})$ of the random walk $\{X_i\}_{i=1}^{\infty}$ on this graph?

Now consider a derived process

$$Y_i = \begin{cases} 0, & \text{if } X_i = 1 \text{ or } 3, \\ 1, & \text{if } X_i = 2. \end{cases}$$

(b) Is it Markov?

(c) Find the entropy rate $H(\mathcal{Y})$ of $\{Y_i\}_{i=1}^{\infty}$.

Now consider another derived process

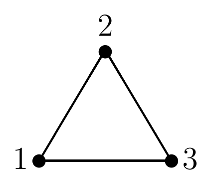
$$Z_i = \begin{cases} 0, & \text{if } X_i = 1 \text{ or } 2, \\ 1, & \text{if } X_i = 3. \end{cases}$$

(d) Is it Markov?

(e) Find the entropy rate $H(\mathcal{Z})$ of $\{Z_i\}_{i=1}^{\infty}$.

For parts (f), (g), and (h), consider the following graph with three nodes:

$\{U_i\}$



(f) What is the entropy rate $H(\mathcal{U})$ of the random walk $\{U_i\}_{i=1}^{\infty}$ on this graph?

Now consider a derived process

$$V_i = \begin{cases} 0, & \text{if } U_i = 1 \text{ or } 2, \\ 1, & \text{if } U_i = 3. \end{cases}$$

(g) Is it Markov?

(h) Find the entropy rate $H(\mathcal{V})$ of $\{V_i\}_{i=1}^{\infty}$.

6. *(20 points)* **Code constraint.**
What is the capacity of a BSC(p) under the constraint that each of the codewords has a proportion of 1's less than or equal to α, i.e.,

$$\frac{1}{n}\sum_{i=1}^{n} X_i(w) \leq \alpha, \qquad \text{for } w \in \{1, 2, \ldots, 2^{nR}\}.$$

(Pay attention when $\alpha > 1/2$.)

7. *(20 points)* **Typicality.**
Let (X, Y) have joint probability mass function $p(x, y)$ given as

$X \backslash Y$	0	1
0	.1	.3
1	.4	.2

(a) Find $H(X), H(Y)$, and $I(X;Y)$. (Don't bother to compute the actual numerical values.)

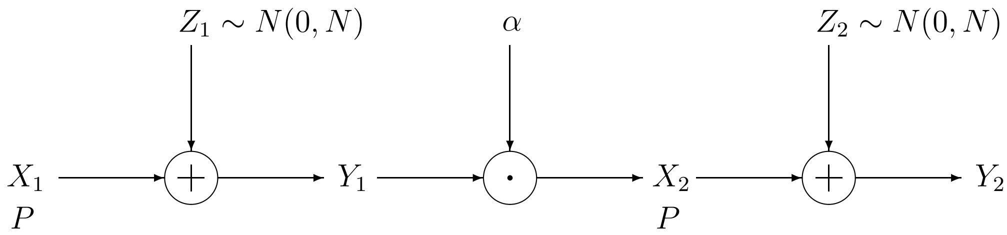
(b) Suppose $\{X_i\}$ is independent and identically distributed (i.i.d.) according to Bern(.4), $\{Y_i\}$ is i.i.d. Bern(1/2), and X^n and Y^n are independent. Find (to first order in the exponent) the probability that (X^n, Y^n) is jointly typical (with respect to the joint distribution $p(x, y)$.

8. *(20 points)* **Partition.**
Let (X, Y) denote height and weight. Let $[Y]$ be Y rounded off to the nearest pound.

(a) Which is greater $I(X;Y)$ or $I(X;[Y])$?

(b) Why?

9. *(20 points)* **Amplify and forward.**
We cascade two Gaussian channels by feeding the (scaled) output of the first channel into the second.



Thus noises Z_1 and Z_2 are independent and identically distributed according to $N(0, N)$,

$$EX_1^2 = EX_2^2 = P,$$

$$Y_1 = X_1 + Z_1,$$
$$Y_2 = X_2 + Z_2,$$

and

$$X_2 = \alpha Y_1,$$

where the scaling factor α is chosen to satisfy the power constraint $EX_2^2 = P$.

(a) *(5 points)* What scaling factor α satisfies the power constraint?

(b) *(10 points)* Find

$$C = \max_{p(x_1)} I(X_1; Y_2).$$

(c) *(5 points)* Is the cascade capacity C greater or less than $\frac{1}{2}\log\left(1 + \frac{P}{N}\right)$?