Homework Set #1
Properties of Entropy and Mutual Information

1. **Entropy of functions of a random variable.**
   Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:

$$\begin{aligned} H(X, g(X)) &\stackrel{(a)}{=} H(X) + H(g(X)|X) \\ &\stackrel{(b)}{=} H(X). \end{aligned}$$

$$\begin{aligned} H(X, g(X)) &\stackrel{(c)}{=} H(g(X)) + H(X|g(X)) \\ &\stackrel{(d)}{\geq} H(g(X)). \end{aligned}$$

   Thus $H(g(X)) \leq H(X)$.

2. **Example of joint entropy.**
   Let $p(x, y)$ be given by

X \\ Y	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

   Find

   (a) $H(X), H(Y)$.
   (b) $H(X|Y), H(Y|X)$.
   (c) $H(X, Y)$.
   (d) $H(Y) - H(Y|X)$.
   (e) $I(X; Y)$.

3. **"True or False" questions**

   Copy each relation and write **true** or **false**. Then, if it's true, prove it. If it is false give a counterexample or prove that the opposite is true.

   (a) $H(X) \geq H(X|Y)$

   (b) $H(X) + H(Y) \leq H(X, Y)$

   (c) Let X, Y be two independent random variables. Then

   $$H(X - Y) \geq H(X).$$

4. **Bytes.**
   The entropy, $H_a(X) = -\sum p(x) \log_a p(x)$ is expressed in bits if the logarithm is to the base 2 and in bytes if the logarithm is to the base 256. What is the relationship of $H_2(X)$ to $H_{256}(X)$?

5. **Two looks.**
   Here is a statement about pairwise independence and joint independence. Let X, Y_1, and Y_2 be binary random variables. If $I(X; Y_1) = 0$ and $I(X; Y_2) = 0$, does it follow that $I(X; Y_1, Y_2) = 0$?

   (a) Yes or no?

   (b) Prove or provide a counterexample.

   (c) If $I(X; Y_1) = 0$ and $I(X; Y_2) = 0$ in the above problem, does it follow that $I(Y_1; Y_2) = 0$? In other words, if Y_1 is independent of X, and if Y_2 is independent of X, is it true that Y_1 and Y_2 are independent?

6. **A measure of correlation.**
   Let X_1 and X_2 be *identically distributed*, but not necessarily independent. Let

   $$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}.$$

   (a) Show $\rho = \frac{I(X_1;X_2)}{H(X_1)}$.

(b) Show $0 \leq \rho \leq 1$.

(c) When is $\rho = 0$?

(d) When is $\rho = 1$?